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Transactions of A. Razmadze Mathematical Institute

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Original article

Approximation in mean on homogeneous compact spaces

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Received 4 August 2016; received in revised form 15 February 2017; accepted 20 February 2017

Available online xxxx

Abstract

Jackson's type theorem on approximation of square integrable functions is proved for functions defined on homogeneous spaces with a compact transitive transformation group actions. An example is proved which illustrates the theorem.
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Keywords: Homogeneous space; Compact group; Quotient group; L^2 space; Fourier series

1. Definition and notations

Let $\mathfrak{W}$ be a homogeneous space, and G be a compact transitive transformation group of $\mathfrak{W}$ with respect to 1 at $\int_G dg = 1$. Let a be a fixed point from $\mathfrak{W}$, and consider its stationary subgroup $H = \{h \in G : ha = a\}$. There exists the following one-to-one correspondence φ between $\mathfrak{W}$ and the quotient space G/H: if $w \in \mathfrak{W}$ and $g \in G$ transforms a to $w \in \mathfrak{W}$, then the corresponding element $\varphi(w) \in G/H$ is the class gH; conversely, the corresponding to a class gH element in $\mathfrak{W}$ is $w = \varphi^{-1}(gH) = ga$. H is a closed subgroup of G and there exists a G-invariant Radon measure μ on G/H, that is, a Radon measure μ such that $\mu(xE) = \mu(E)$ for every $x \in G$, $E \subset G/H$. ([1], 2.49–2.53, 2.7). μ is unique up to a constant factor, and if this factor is suitably chosen, then

$$\int_G f(g)dg = \int_{G/H} \int_H f(g\xi)d\xi d\mu(gH) \tag{1}$$

for any function f integrable on G with respect to the Haar measure.

A representation T is of class 1 with respect to H, if its carrier space $\mathcal{L}$ contains non-zero vectors invariant with respect to all operators Th, $h \in H$ and all these operators are unitary [2, p. 103]. In what follows, A_H will stand for a set of indices for which $(T_l)_{l \in A_H}$ is the family of all pairwise nonequivalent irreducible representations of G which

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Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.02.003
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Please cite this article in press as: D. Ugulava, Approximation in mean on homogeneous compact spaces., Transactions of A. Razmadze Mathematical Institute (2017), http://dx.doi.org/10.1016/j.trmi.2017.02.003

are of class 1 with respect to H. Below, we consider the case when the stationary group H is massive. This means that, for any representation of the group G of class 1 with respect to H, the subspace of vectors in $\mathcal{L}_l$, invariant for H, is one-dimensional ([2], p. 103). Let $\{e_i^l\}$ be an orthonormal basis of $\mathcal{L}_l$ such that $T_l(e_1^l) = e_1^l$.

A measure $d_{\mathfrak{W}}$ on $\mathfrak{W}$ can be introduced by a G-invariant Radon measure $d_{G/H}$ of the compact quotient space G/H as follows: if the set $\varphi(W) \subset G/H$ corresponds to a set $W \subset \mathfrak{W}$, then their measures are equal to each other. Every function f given on $\mathfrak{W}$ with a transformation group G can be regarded as a function on G which is constant on the left cosets with respect to the stationary subgroup H with respect to a point $a \in \mathfrak{W}$. Namely, if the class gH corresponds to a $w \in \mathfrak{W}$, then the function $f_a(g) = f_a(gH) = f(w)$ defined on G corresponds to a function $f(w)$. We say that a defined on $\mathfrak{W}$ function $f(w)$ belongs to the space $L^2(\mathfrak{W})$, if $f_a(\varphi(w)) \in L^2(G/H)$ or $f_a(g) \in L^2(G)$. Conversely, if a function $f_a \in L^2(G)$ is constant on the left cosets with respect to H, then its corresponding function $f(w)$ belongs to $L^2(\mathfrak{W})$. $L^2(\mathfrak{W})$ is a Hilbert space with respect to the usually norm. It is clear that an expansion of a function $f \in L^2(\mathfrak{W})$ on a homogeneous space $\mathfrak{W}$ can be obtained by means of the Fourier expansion of $f_a \in L^2(G)$, which is invariant under left shifts by elements of the corresponding to $a \in \mathfrak{W}$ subgroup H.

Let us denote by $L_H^2(G)$ the subspace in $L^2(G)$ which is invariant under left shifts by elements of the massive stationary subgroup H of a fixed $a \in \mathfrak{W}$. Any function f_a from $L_H^2(G)$ can be expanded into the Fourier series of the form ([2], p. 105)

$$f_a(g) = \sum_{l \in A_H} Y_l(f_a, g), \quad \text{where} \quad Y_l(f_a, g) = \sum_{m=1}^{d_l} c_m^l t_{m1}^l(g). \tag{2}$$

The integer d_l in (2) is the dimension of the carrier space $\mathcal{L}_l$ of the representation $T_l(g)$ and $t_{m1}^l(g)$, $1 \le m \le d_l$ are the matrix functions of the representation T_l. Together with $f_a(g)$, the matrix functions $t_{m1}^l(g)$ are also invariant under left shifts by elements of the subgroup H ([2], p. 104). The coefficients c_m^l, according to (1), are given by the equalities

$$\begin{aligned} c_m^l &= d_l \int_G f_a(g)\overline{t_{m1}^l(g)}dg = d_l \int_{G/H}\int_H f_a(g\xi)\overline{t_{m1}^l(g\xi)}d\xi d\mu(gH) \\ &= d_l \mathrm{mes} H \int_{G/H} f_a(u)\overline{t_{m1}^l(u)}d_\mu(u) = d_l \mathrm{mes} H \int_{\mathfrak{W}} f(w)\overline{t_{m1}^l(\varphi(w))}dw, \end{aligned} \tag{3}$$

where $\mathrm{mes} H$ is Haar measure of subgroup $H \subset G$.

The index l in (2) becomes a countable number of values, for which $c_m^l \neq 0$. List them as $\{l_1, \dots, l_n, \dots\}$. Following [3], the symbol $l < n$ is interpreted as $l \in \{l_1, \dots, l_n\}$, and $l \ge n$ denotes that $l \in A_H \setminus \{l_1, \dots, l_n\}$.

Thus, if $f \in L^2(\mathfrak{W})$ and H is a massive stationary subgroup of an element $a \in \mathfrak{W}$ in G, then f can be expanded in the series of the form

$$f(w) = \sum_{l \in A_H} \sum_{m=1}^{d_l} c_m^l t_{m1}^l(\varphi(w)),$$

where the coefficients c_m^l are defined by (3).

If

$$S_n(f, w) = \sum_{l \in A_H, l \le n} \sum_{m=1}^{d_l} c_m^l t_{m1}^l(\varphi(w)), n \in \mathbb{N},$$

is the nth partial sum of this series, and $S_n(f_a, g) - n$th partial sum of the series (2), then

$$f(w) - S_n(f, w) = f_a(g) - \sum_{l \in A_H;\ l \le n} Y_l(f_a, g). \tag{4}$$

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