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Original article

Higher-order commutators of parametrized Littlewood–Paley operators on Herz spaces with variable exponent

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Abstract

Let $\Omega \in L^2(S^{n-1})$ be a homogeneous function of degree zero and b be a BMO or Lipschitz function. In this paper, we obtain some boundedness of the parametrized Littlewood–Paley operators and their high-order commutators on Herz spaces with variable exponent.

Keywords: Herz space; Variable exponent; Commutator; Parametrized area integral; Parametrized Littlewood–Paley g_λ^* function

1. Introduction

The theory of function spaces with variable exponent has been extensively studied by researchers since the work of Kováčik and Rákosník [1] appearing in 1991. In [2–5] and [6], the authors proved the boundedness of some integral operators on variable L^p spaces.

Given an open set $E \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : E \longrightarrow [1, \infty)$, $L^{p(\cdot)}(E)$ denotes the set of measurable functions f on E such that for some $\lambda > 0$,

$$\int_E \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg–Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\lambda > 0 : \int_E \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

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These spaces are referred to as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(E)$ is isometrically isomorphic to $L^p(E)$.

The space $L^{p(\cdot)}_{\mathrm{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(E) := \{f : f \in L^{p(\cdot)}(F) \text{ for all compact subsets } F \subset E\}.$$

Define $\mathcal{P}(E)$ to be the set of $p(\cdot) : E \longrightarrow [1, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in E\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in E\} < \infty.$$

Denote $p'(x) = p(x)/(p(x) - 1)$.

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy,$$

where $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbb{R}^n$ by $|A|$ and χ_A, respectively.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1.1. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfies*

$$|p(x) - p(y)| \le \frac{C}{-\log(|x - y|)}, \quad |x - y| \le 1/2 \tag{1.1}$$

and

$$|p(x) - p(y)| \le \frac{C}{\log(|x| + e)}, \quad |y| \ge |x|, \tag{1.2}$$

then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, that is the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 1.2 ([1]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is called the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.3 ([4]). *Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1} \text{ and } \frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Throughout this paper δ_1 and δ_2 are the same as in Lemma 1.3.

Lemma 1.4 ([4]). *Suppose $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C.$$

Next we recall the definition of the Herz-type spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of all positive and non-negative integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$.