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Original article

On Robinson's Energy Delay Theorem

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Abstract

An elementary proof of Robinson's Energy Delay Theorem on minimum-phase functions is provided. The situation in which the energy conservation property holds for an infinite number of lags is fully described.

Keywords: Minimum-phase functions; Hardy spaces

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane and $\mathbb{T} = \{z \in \mathbb{C}: |z| = 1\}$ be its boundary. The set of all analytic in $\mathbb{D}$ functions is denoted by $\mathcal{A}(\mathbb{D})$. The Hardy space $H^2 = H^2(\mathbb{D})$ consists of all the functions $f \in \mathcal{A}(\mathbb{D})$ the Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

of which satisfy the condition

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

In engineering, these functions are known as *z-transforms* (resp. *transfer functions*) of discrete-time *causal signals* (resp. *filter impulse responses*) with a finite energy. It is well known that the boundary values of $f \in H^2$ exist a.e.,

$$f_+(e^{i\theta}) = \lim_{r\to 1-} f(re^{i\theta}) \quad \text{for a.a. } \theta \in [0, 2\pi), \tag{1}$$

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and $f_+ \in L^2(\mathbb{T})$, the Lebesgue space of square integrable functions on $\mathbb{T}$. Furthermore, $f_+ \in L^2_+(\mathbb{T}) := \{f \in L^2(\mathbb{T}) : c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,d\theta = 0 \text{ for } n < 0\}$. Actually, there is a one-to-one correspondence between H^2 and $L^2_+(\mathbb{T})$, and therefore we may naturally identify these two classes.

For any function $f \in H^2$, the inequality

$$|f(0)| \le \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f_+(e^{i\theta})|\,d\theta\right) \tag{2}$$

holds (see, e.g., [1, Th. 17.17]). The extreme functions for which (2) turns into an equality are called *outer*. In engineering they are also known as *minimum-phase*, or *optimal*, functions. According to the original definition of outer functions by Beurling [2], they admit the representation

$$f(z) = c \cdot \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f_+(e^{i\theta})|\,d\theta\right), \tag{3}$$

where c is a unimodular constant. This representation easily implies that the equality holds in (2) for outer functions and it can be proved that the converse is also true. In particular, boundary values of the modulus of an outer function uniquely determine the function itself up to a constant multiple with absolute value 1.

The following property of minimum-phase functions, first observed by Robinson [3], plays an important role in several signal processing applications.

Theorem 1. *Let $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$ be H^2-functions satisfying*

$$|f_+(e^{i\theta})| = |g_+(e^{i\theta})| \text{ for a.e. } \theta. \tag{4}$$

If f is of minimum-phase, then for each N,

$$\sum_{n=0}^N |a_n|^2 \ge \sum_{n=0}^N |b_n|^2. \tag{5}$$

Robinson gave a physical interpretation to inequality (5) "that among all filters with the same gain, the outer filter makes the energy built-up as large as possible, and it does so for every positive time" [4] and found geological applications of minimum-phase waveforms. Consequently, the term *minimum-delay* [5, p. 211] functions is being used to describe optimal functions, and Theorem 1 is known as the Energy Delay Theorem within the geological community [6, p. 52].

Theorem 1 was further extended to the matrix polynomial case and used in MIMO communications in [7]. In [8], the theorem is formulated and proved for general operator valued functions in abstract Hilbert spaces.

In this paper, we provide a very short and simple proof of Theorem 1 based on classical facts from the theory of Hardy spaces. This is done in Section 3, while the modification of this proof fitting the matrix case is discussed in Section 4. In final Section 5, we treat the situation in which (5) turns into an equality for infinitely many values of N. The preliminary Section 2 contains some notation and known results, included for convenience of reference.

2. Notation

Let $L^p = L^p(\mathbb{T})$, $0 < p \le \infty$, be the Lebesgue space of p-integrable complex functions f with the norm $\|f\|_{L^p} = \left(\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta\right)^{\frac{1}{p}}$ for $p \ge 1$ (with the standard modification for $p = \infty$), and let $H^p = H^p(\mathbb{D})$, $0 < p \le \infty$, be the Hardy space

$$\left\{ f \in \mathcal{A}(\mathbb{D}) : \sup_{r<1}\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta < \infty \right\}$$

with the norm $\|f\|_{H^p} = \sup_{r<1}\|f(re^{i\cdot})\|_{L^p}$ for $p \ge 1$ (H^∞ is the space of bounded analytic functions with the supremum norm). It is well known that boundary value function f_+ (see (1)) exists for every $f \in H^p$, $p > 0$, and