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Original article

Sharp weighted bounds for the Hilbert transform of odd and even functions

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Abstract

Our aim is to establish sharp weighted bounds for the Hilbert transform of odd and even functions in terms of the mixed type characteristics of weights. These bounds involve A_p and A_∞ type characteristics. As a consequence, we obtain weighted bounds in terms of so-called Andersen–Muckenhoupt type characteristics.

Keywords: Hilbert transform; Sharp weighted bound; One-weight inequality

1. Introduction

In this paper, we investigate sharp weighted bounds, involving A_p and A_∞ characteristics of weights, for the Hilbert transform of odd and even functions. Following general results we derive these sharp weighted A_p bounds in terms of so-called Andersen-Muckenhoupt characteristics. Let X and Y be two Banach spaces. Given a bounded operator $T : X \to Y$, we denote the operator norm by $\|T\|_{\mathcal{B}(X,Y)}$ which is defined in the standard way i.e. $\|T\|_{\mathcal{B}(X,Y)} = \sup_{\|f\|_X \le 1} \|Tf\|_Y$. If $X = Y$ we use the symbol $\|T\|_{\mathcal{B}(X)}$.

A non-negative locally integrable function (i.e. a weight function) w defined on $\mathbb{R}^n$ is said to satisfy the $A_p(\mathbb{R}^n)$ condition ($w \in A_p(\mathbb{R}^n)$) for $1 < p < \infty$ if

$$\|w\|_{A_p(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} < \infty,$$

where $p' = \frac{p}{p-1}$ and supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes. We call $\|w\|_{A_p(\mathbb{R}^n)}$ the A_p characteristic of w.

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In 1972, B. Muckenhoupt [1] showed that if $w \in A_p(\mathbb{R}^n)$, where $1 < p < \infty$, then the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|dy$$

is bounded in $L^p_w(\mathbb{R}^n)$. S. Buckley [2] investigated the sharp A_p bound for the operator M. In particular, he established the inequality

$$\|M\|_{L^p_w(\mathbb{R}^n)} \le C\|w\|^{\frac{1}{p-1}}_{A_p(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{1.1}$$

Moreover, he showed that the exponent $\frac{1}{p-1}$ is best possible in the sense that we cannot replace $\|w\|^{\frac{1}{p-1}}_{A_p}$ by $\psi(\|w\|_{A_p})$ for any positive non-decreasing function ψ growing slowly than $x^{\frac{1}{p-1}}$. From here it follows that for any $\lambda > 0$,

$$\sup_{w \in A_p} \frac{\|M\|_{L^p_w}}{\|w\|^{\frac{1}{p-1}-\lambda}_{A_p}} = \infty.$$

Let H be the Hilbert transform given by

$$(Hf)(x) = p.v.\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x-t}dt, \quad x \in \mathbb{R}.$$

In 1973 R. Hunt, B. Muckenhoupt and R. L. Wheeden [3] solved the one-weight problem for the Hilbert transform in terms of Muckenhoupt condition. In particular, they established the inequality

$$\|Hf\|_{L^p_w(\mathbb{R})} \le c_p \|w\|^{\beta}_{A_p(\mathbb{R})} \|f\|_{L^p_w(\mathbb{R})} \tag{1.2}$$

for some positive constant β and some constant c_p depending on p. S. Petermichl showed that the value of the exponent $\beta = \max\{1, p'/p\}$ in (1.2) is sharp. In particular, the following statement holds (see [4] for $p = 2$, [5] for $p \neq 2$):

Theorem A. *Let $1 < p < \infty$ and let w be a weight function on $\mathbb{R}$. Then there is a positive constant c_p depending only on p such that*

$$\|H\|_{\mathcal{B}(L^p_w)} \le c_p \|w\|^{\beta}_{A_p(\mathbb{R})}, \tag{1.3}$$

where $\beta = \max\left\{1, \frac{p'}{p}\right\}$. Moreover, the exponent in (1.3) is sharp.

We say that $w \in A_\infty(\mathbb{R}^n)$ if $w \in A_p(\mathbb{R})$ for some $p > 1$. In what follows we will use the symbol $\|\rho\|_{A_\infty}$ for the A_∞ characteristic of a weight function ρ:

$$\|\rho\|_{A_\infty} = \sup_I \frac{1}{\rho(I)} \int_I M(\rho\chi_I)(x)dx.$$

This characteristic appeared first in the papers by Fiji [6] and Wilson [7,8] and is lower than that the one introduced by Hruščev [9]:

$$[\rho]_{A_\infty} = \sup_I \left(\frac{1}{|I|} \int_I \rho(x)dx\right) \exp\left(\frac{1}{|I|} \int_I \log \rho^{-1}(x)dx\right).$$

In 2012, Hytönen, Perez and Rela [10] improved Buckley's result and obtained a sharp weighted bound involving A_∞ constant:

$$\|M\|_{\mathcal{B}(L^p_w)} \le c_n \left(\frac{1}{p-1} \|w\|_{A_p} \|\sigma\|_{A_\infty}\right)^{1/p}, \quad 1 < p < \infty, \quad \sigma = w^{1-p'}.$$

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