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The maximum relaxation time of a random walk

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We show the minimum spectral gap of the normalized Laplacian over all simple, connected graphs on n vertices is $(1 + o(1))\frac{54}{n^3}$. This minimum is achieved asymptotically by a double kite graph. Consequently, this leads to sharp upper bounds for the maximum relaxation time of a random walk, settling a conjecture of Aldous and Fill. We also improve an eigenvalue–diameter inequality by giving a new lower bound for the spectral gap of the normalized Laplacian. This eigenvalue lower bound is asymptotically best possible.

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1. Introduction

Graph eigenvalues play a powerful role in the study of random walks. In particular, eigenvalues are a primary tool for bounding a number of key random walk parameters, such as mixing time. Consequently, bounds on graph eigenvalues are not only of interest in themselves, but also may have immediate implications for the behavior of the random walk (for a survey, see [14]). In the case of the *relaxation time* of a discrete reversible Markov chain, eigenvalues themselves define the quantity of interest.

In this paper, we examine an extremal problem concerning the normalized Laplacian spectral gap, the reciprocal of which defines the relaxation time of a random walk. The normalized Laplacian matrix $\mathcal{L}$ of a graph G is

$$\mathcal{L} = I - T^{-1/2} A T^{-1/2},$$

where T denotes the diagonal degree matrix with (u,u) entry equal to $d(u)$ and A denotes the adjacency matrix. Throughout, we assume G is simple, meaning G has no loops or multiple edges. We write the eigenvalues of $\mathcal{L}$ in increasing order, where

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq 2.$$

It is well-known (cf. [6]) that the second eigenvalue or spectral gap of $\mathcal{L}$ is nonzero if and only if G is connected, and can be characterized as

$$\lambda_1 = \inf_{\substack{f \\ \sum_u f(u)d(u)=0}} \frac{\displaystyle\sum_{u \sim v} (f(u) - f(v))^2}{\displaystyle\sum_v f(v)^2 d(v)},$$

with corresponding eigenvector $g = T^{1/2} f$. We call the nontrivial function f achieving the above infimum the *harmonic eigenfunction* of $\mathcal{L}$. Landau and Odlyzko proved the following lower bound on λ_1.

Theorem 1 *(Landau, Odlyzko [12]). For a connected graph on n vertices with maximum degree Δ and diameter D, we have*

$$\lambda_1 \geq \frac{1}{n\Delta(D+1)}.$$

In [6], Chung gives an improved lower bound on λ_1 in terms of the graph's diameter and volume, where $\mathrm{vol}(G) = \sum_{u \in V(G)} d(u)$.

Theorem 2 *(Chung [6]). For a connected graph G with diameter D, we have*

$$\lambda_1 \geq \frac{1}{D \cdot \mathrm{vol}(G)}.$$