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On the structure of matrices avoiding interval-minor patterns

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We study the structure of 01-matrices avoiding a pattern P as an interval minor. We focus on critical P-avoiders, i.e., on the P-avoiding matrices in which changing a 0-entry to a 1-entry always creates a copy of P as an interval minor.

Let Q be the 3×3 permutation matrix corresponding to the permutation 231. As our main result, we show that for every pattern P that has no rotated copy of Q as interval minor, there is a constant c_P such that any row and any column in any critical P-avoiding matrix can be partitioned into at most c_P intervals, each consisting entirely of 0-entries or entirely of 1-entries. In contrast, for any pattern P that contains a rotated copy of Q, we construct critical P-avoiding matrices of arbitrary size $n \times n$ having a row with $\Omega(n)$ alternating intervals of 0-entries and 1-entries.

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1. Introduction

A *binary matrix* is a matrix with entries equal to 0 or 1. All matrices considered in this paper are binary. The study of extremal problems of binary matrices has been initiated by the papers of Bienstock and Győri [1] and of Füredi [7]. Since these early works, most of the research in this area has focused on the concept of forbidden submatrices: a matrix M is said to contain a pattern P as a submatrix if we can transform M into P by deleting some rows and columns, and by changing 1-entries into 0-entries. This notion of submatrix is a matrix analogue of the notion of subgraph in graph theory.

The main problem in the study of pattern-avoiding matrices is to determine the extremal function $\mathrm{ex}(n; P)$, defined as the largest number of 1-entries in an $n \times n$ binary matrix avoiding the pattern P as submatrix. This is an analogue of the classical Turán-type problem of finding a largest number of edges in an n-vertex graph avoiding a given subgraph. Despite the analogy, the function $\mathrm{ex}(n; P)$ may exhibit an asymptotic behaviour not encountered in Turán theory. For instance, for the pattern[1] $P = \begin{pmatrix} \bullet & & \bullet & \\ & \bullet & & \bullet \end{pmatrix}$ Füredi and Hajnal [8] proved that $\mathrm{ex}(n; P) = \Theta(n\alpha(n))$, where $\alpha(n)$ is the inverse of the Ackermann function.

The asymptotic behaviour of $\mathrm{ex}(n; P)$ for general P is still not well understood. Füredi and Hajnal [8] posed the problem of characterising the *linear* patterns, i.e., the patterns P satisfying $\mathrm{ex}(n; P) = O(n)$. Marcus and Tardos [15] proved that $\mathrm{ex}(n; P) = O(n)$ whenever P is a *permutation matrix*, i.e., P has exactly one 1-entry in each row and each column. This result, combined with previous work of Klazar [12], has confirmed the long-standing Stanley–Wilf conjecture. However, the problem of characterising linear patterns is still open despite a number of further partial results [3,6,9,11,17,19].

Fox [5] has introduced a different notion of containment among binary matrices, based on the concept of interval minors. Informally, a matrix M contains a pattern P as an interval minor if we can transform M into P by contracting adjacent rows or columns and changing 1-entries into 0-entries; see Section 2 for the precise definition. In this paper, we mostly deal with containment and avoidance of interval minors rather than submatrices. Therefore, the phrases M *avoids* P or M *contains* P always refer to avoidance or containment of interval minors, and the term P*-avoider* always refers to a matrix that avoids P as interval minor.

In analogy with $\mathrm{ex}(n; P)$, it is natural to consider the corresponding extremal function $\mathrm{ex}_{\preccurlyeq}(n; P)$ as the largest number of 1-entries in an $n \times n$ matrix that avoids P as an interval minor. If M contains P as a submatrix, it also contains it as an interval minor, and therefore $\mathrm{ex}_{\preccurlyeq}(n; P) \leq \mathrm{ex}(n; P)$. Moreover, it can be easily seen that for a permutation matrix P the two notions of containment are equivalent, and hence $\mathrm{ex}_{\preccurlyeq}(n; P) = \mathrm{ex}(n; P)$.

Fox [5] used interval minors as a key tool in his construction of permutation patterns with exponential Stanley–Wilf limits. In view of the results of Cibulka [2], this is equiv-

[1] We use the convention of representing 1-entries in binary matrices by dots and 0-entries by blanks.