Advances in Applied Mathematics 101 (2018) 168–183

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Fixed points and adjacent ascents for classical complex reflection groups

Francesco Brenti [a], Mario Marietti [b,*]

[a] *Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 1, 00133 Roma, Italy*

[b] *Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, Italy*

A R T I C L E I N F O

Article history:
Received 3 June 2018
Received in revised form 27 July 2018
Accepted 1 August 2018
Available online xxxx

MSC:
primary 05A05
secondary 05E15, 20F55

A B S T R A C T

We characterize the classical complex reflection groups for which a recent symmetric group equidistribution result studied by Diaconis, Evans, and Graham holds. This leads to some refinements of the original result, which seem to be new even in the symmetric group case.

1. Introduction

In a recent paper [5] Diaconis, Evans, and Graham give various proofs of the fact that the number of permutations in S_n with a given set I of fixed points distinct from n is equal to the number of permutations in S_n having I as a set of unseparated pairs, for all $I \subseteq [n-1]$ (see §2 Theorem 2.2 for the relevant definitions).

This result is equivalent to the fact that the two sets of permutations $A = \{\sigma \in S_n : \{k \in [n-1] : \sigma(k+1) = \sigma(k)+1\} \supseteq I\}$ and $B = \{\sigma \in S_n : \{k \in [n-1] : \sigma(k) = k\} \supseteq I\}$

* Corresponding author.
E-mail addresses: brenti@mat.uniroma2.it (F. Brenti), m.marietti@univpm.it (M. Marietti).

https://doi.org/10.1016/j.aam.2018.08.001
0196-8858/

have the same cardinality for all $I \subseteq [n-1]$. In this paper, we first show that these two sets are just the two "extreme cases" of a large class of subsets of S_n, all having the same cardinality: these subsets of permutations are indexed by a pair (J, K) of subsets of $[n-1]$ and have cardinalities that depend only on $|J| + |K|$. The two subsets A and B are obtained as special cases with the indexing pairs $(\emptyset, I)$ and $(I, \emptyset)$. We believe that this result sheds some new light on Theorem 2.2. We then investigate the extent to which Theorem 2.2 continues to hold for the classical complex reflection groups, which are the colored permutation groups $G(r, p, n)$, where $r, p, n \in \mathbb{P}$ and $p \,|\, r$ (see §2 for the definition). More precisely, we show that, with suitable definition of "unseparated pair" (which we prefer to call "adjacent ascent"), the result holds for the group $G(r, p, n)$ if and only if $\gcd(p, n) = 1$ (Corollary 4.2). Our investigation naturally leads to a refinement of the result in [5] (Corollary 4.8), which, in turn, holds for $G(r, p, n)$ if and only if all divisors of p are larger than n (Corollary 4.3). We also investigate some other possible definitions of adjacent ascent for colored permutations and the corresponding equidistribution results. Our proofs are enumerative and combinatorial.

The organization of the paper is as follows. In the next section, we recall some definitions, notation, and results that are used in what follows. In §3, we introduce the concept of compatible subsets and obtain a result that includes the one in [5] as the two "extreme cases". In §4, we propose a definition of adjacent ascent for any colored permutation and characterize the groups $G(r, p, n)$ for which the result in [5] holds. This naturally leads to a refinement of the original theorem, and we characterize the groups $G(r, p, n)$ for which this refinement holds. This refinement seems to be new even in the symmetric group case. In §5, we investigate two other possible definitions of "adjacent ascent" for colored permutations and the corresponding equidistribution results. In §6, we provide bijective proofs of two of the results in the previous sections.

2. Preliminaries

In this section we recall some notation, definitions, and results that are used in what follows.

We let $\mathbb{Z}$, $\mathbb{P}$, and $\mathbb{N}$ be the set of integers, positive integers, and nonnegative integers, respectively. Given $n, m \in \mathbb{N}$, with $n \le m$, we let $[n, m] := \{n, n+1, \ldots, m\}$ and, for $n \in \mathbb{P}$, we let $[n] := [1, n]$. For a set T, we let $S(T)$ be the set of all bijections of T, and $S_n := S([n])$.

Let $n \in \mathbb{P}$. By a *composition* of n we mean a sequence $\alpha = (\alpha_1, \ldots, \alpha_s)$ (for some $s \in \mathbb{P}$) of positive integers such that $\sum_{i=1}^{s} \alpha_i = n$. We let $\mathcal{C}_n$ be the set of all compositions of n, and $\mathcal{C} := \bigcup_{n \ge 1} \mathcal{C}_n$. Given $\beta, \alpha \in \mathcal{C}_n$, with $\beta = (\beta_1, \ldots, \beta_s)$ and $\alpha = (\alpha_1, \ldots, \alpha_t)$, we say that β *refines* α if there exist $1 \le i_1 < i_2 < \cdots < i_{t-1} < s$ such that $\sum_{j=i_{k-1}+1}^{i_k} \beta_j = \alpha_k$ for $k = 1, \ldots, t$ (where $i_0 := 0$, $i_t := s$). We then write $\beta \preceq \alpha$. Moreover, for $\beta = (\beta_1, \ldots, \beta_s) \in \mathcal{C}_n$, we let $\pi(\beta)$ be the non-increasing rearrangement of $\beta_1, \ldots, \beta_s$. For any other undefined notation and terminology in enumerative combinatorics, we follow [10] and [9].