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Restricted growth function patterns and statistics ☆

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A restricted growth function (RGF) of length n is a sequence $w = w_1w_2 \dots w_n$ of positive integers such that $w_1 = 1$ and $w_i \le 1 + \max\{w_1, \dots, w_{i-1}\}$ for $i \ge 2$. RGFs are of interest because they are in natural bijection with set partitions of $\{1, 2, \dots, n\}$. An RGF w avoids another RGF v if there is no subword of w which standardizes to v. We study the generating functions $\sum_{w \in R_n(v)} q^{\mathrm{st}(w)}$ where $R_n(v)$ is the set of RGFs of length n which avoid v and $\mathrm{st}(w)$ is any of the four fundamental statistics on RGFs defined by Wachs and White. These generating functions exhibit interesting connections with multiset permutations, integer partitions,

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Noncrossing partition
Nonnesting partition
Pattern
rb
rs
Restricted growth function
Partition
Statistic
Two-colored Motzkin path

and two-colored Motzkin paths, as well as noncrossing and nonnesting set partitions.

1. Introduction

Recently, there has been a flurry of activity looking at the distribution of statistics over pattern classes in various objects. For example, see [4,5,7,9,11,12,14]. There are two notions of pattern containment for set partitions, one obtained by standardizing a subpartition and one obtained by standardizing a subword of the corresponding restricted growth function. In [6], the present authors studied the distribution of four fundamental statistics of Wachs and White [25] over avoidance classes using the first definition. The purpose of this paper is to carry out an analogous investigation for the second.

Let us begin by defining our terms. Consider a finite set S. A *set partition* σ of S is a family of nonempty subsets $B_1, \ldots, B_k$ whose disjoint union is S, written $\sigma = B_1/\ldots/B_k \vdash S$. The B_i are called *blocks* and we will usually suppress the set braces and commas in each block for readability. We will be particularly interested in set partitions of $[n] := \{1, 2, \ldots, n\}$ and will use the notation

$$\Pi_n = \{\sigma \ : \ \sigma \vdash [n]\}.$$

To illustrate $\sigma = 145/2/3 \vdash [5]$. If $T \subseteq S$ and $\sigma = B_1/\ldots/B_k \vdash S$ then there is a corresponding *subpartition* $\sigma' \vdash T$ whose blocks are the nonempty intersections $B_i \cap T$. To continue our example, if $T = \{2, 4, 5\}$ then we get the subpartition $\sigma' = 2/45 \vdash T$.

The concept of pattern is built on the standardization map. Let O be an object with labels which are positive integers. The *standardization* of O, $\mathrm{stan}(O)$, is obtained by replacing all occurrences of the smallest label in O by 1, all occurrences of the next smallest by 2, and so on. Say that $\sigma \vdash [n]$ *contains* π *as a pattern* if it contains a subpartition σ' such that $\mathrm{stan}(\sigma') = \pi$. In this case σ' is called an *occurrence* or *copy* of π in σ. Otherwise, we say that σ *avoids* π and let

$$\Pi_n(\pi) = \{\sigma \in \Pi_n \ : \ \sigma \text{ avoids } \pi\}.$$

In our running example, $\sigma = 145/2/3$ contains $\pi = 1/23$ since $\mathrm{stan}(2/45) = 1/23$. But σ avoids $12/3$ because if one takes any two elements from the first block of σ then it is impossible to find an element from another block bigger than both of them. Klazar [16–18] was the first to study this approach to set partition patterns. For more recent work, see the paper of Bloom and Saracino [3].