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Simplicial networks and effective resistance

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A B S T R A C T

We introduce the notion of effective resistance for a *simplicial network* (X, R) where X is a simplicial complex and R is a set of resistances for the top simplices, and prove two formulas generalizing previous results concerning effective resistance for resistor networks. Our approach, based on combinatorial Hodge theory, is to assign a unique harmonic class to a *current generator* σ, an extra top-dimensional simplex to be attached to X. We will show that the harmonic class gives rise to the *current* I_σ and the *voltage* V_σ for $X \cup \sigma$, satisfying Thomson's energy-minimizing principle and Ohm's law for simplicial networks.

The effective resistance R_σ of a current generator σ shall be defined as a ratio of the σ-components of V_σ and I_σ. By introducing *potential* for voltage vectors, we present a formula for R_σ via the inverse of the weighted combinatorial Laplacian of X in codimension one. We also derive a formula for R_σ via weighted high-dimensional tree-numbers for X, providing a combinatorial interpretation for R_σ. As an application, we generalize Foster's Theorem, and discuss various high-dimensional examples.

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1. Introduction

A *simplicial network* (X, R) consists of a simplicial complex X of dimension d (> 0) and a set R of positive *resistances* for the d-dimensional simplices of X. Additional topological conditions for X will be assumed later as needed. A *current generator* σ is a d-dimensional simplex that is attached to X resulting in a (cell) complex $Y = X \cup \sigma$. The purpose of this paper is to introduce the notion of *effective resistance* R_σ of a current generator σ, and present its formulas and applications. Simplicial networks are a generalization of resistor networks, and the current work aims to extend classical results (see *e.g.* [15,19]) concerning effective resistance for resistor networks.

Let us outline our approach to R_σ. Suppose a nonzero real number i_σ is assigned to a current generator σ. We will associate a unique cycle I_σ in the chain group $\mathcal{C}_d(Y; \mathbb{R})$, which we call the *current vector* induced by i_σ, as follows. Attach σ to an *acyclization* $\mathcal{A}(X)$ of X (see Section 2) to form a complex Z with rank 1 homology group in dimension d (see Section 3 for the definition of Z). By combinatorial Hodge theory [9], there is a unique *harmonic class* for Z determined by i_σ. This harmonic class is the desired I_σ when every element in R equals 1. Otherwise, a similar argument using a *weighted* chain complex for Z will produce I_σ (see Section 3.2). As we shall see, the *energy-minimizing* property of a harmonic class is a high-dimensional analogue of Thomson's Principle for currents in a resistor network. Also, we will define a *voltage vector* $V_\sigma \in \mathcal{C}_d(Y; \mathbb{R})$ by requiring Ohm's law [4] and the orthogonality of current and voltage vectors. In short, we have the current I_σ and voltage V_σ vectors for $Y = X \cup \sigma$ uniquely determined by a given nonzero current i_σ through σ. Now, we shall define R_σ as a ratio of the respective σ-components v_σ and i_σ of V_σ and I_σ.

We will present another definition of R_σ by introducing *potential* for voltage vectors (see (11) in Section 4). For a 1-dimensional potential theory, refer to [3]. Using this definition, we obtain a formula for R_σ via the inverse of the weighted combinatorial Laplacian for X in codimension 1 where the weights are given by the conductances $C = R^{-1}$ (regarding R as a diagonal matrix). This formula generalizes that of effective resistance for 1-dimensional networks via the inverse of the combinatorial Laplacian in dimension zero [18,15].

We will obtain another formula for R_σ (Theorem 5.2) via weighted high-dimensional tree-numbers for X with the weights $C = R^{-1}$. We refer the readers to [5,6,13] for high-dimensional tree-numbers. This formula generalizes a well-known combinatorial interpretation [19] of effective resistance for resistor networks. For its application, we will derive a high-dimensional analogue of Foster's Theorem [8], and compute effective resistance for the standard simplexes (Example 5.5), the complete colorful complexes (Example 5.6), and the hypercubes (Example 5.7).

2. Preliminaries

In this section, we will review definitions regarding simplicial complexes and homology groups. Refer to [17] for further details. We will collect relevant definitions and