Advances in Applied Mathematics 100 (2018) 101–121

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Quasisymmetric and noncommutative skew Pieri rules

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A R T I C L E I N F O

Article history:
Received 19 July 2017
Received in revised form 29 May 2018
Accepted 29 May 2018
Available online xxxx

MSC:
primary 05E05, 16T05, 16W55
secondary 05A05, 05E10

Keywords:
Composition
Composition poset
Composition tableau
Noncommutative symmetric function
Quasisymmetric function
Skew Pieri rule

A B S T R A C T

In this note we derive skew Pieri rules in the spirit of Assaf–McNamara for skew quasisymmetric Schur functions using the Hopf algebraic techniques of Lam–Lauve–Sottile, and recover the original rules of Assaf–McNamara as a special case. We then apply these techniques a second time to obtain skew Pieri rules for skew noncommutative Schur functions.

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https://doi.org/10.1016/j.aam.2018.05.005

1. Introduction

The Hopf algebra of quasisymmetric functions, QSym, was first defined explicitly in [11]. It is a nonsymmetric generalization of the Hopf algebra of symmetric functions, and arises in many areas such as the representation theory of the 0-Hecke algebra [4, 7,17,34], probability [14,31], and is the terminal object in the category of combinatorial Hopf algebras [1]. Recently a basis of QSym, known as the basis of quasisymmetric Schur functions, was discovered [13], which is a nonsymmetric generalization of the symmetric function basis of Schur functions. These quasisymmetric Schur functions arose from the combinatorics of Macdonald polynomials [12], have been used to resolve the conjecture that QSym over the symmetric functions has a stable basis [19], and have initiated the dynamic research area of discovering other quasisymmetric Schur-like bases such as row-strict quasisymmetric Schur functions [9,24], Young quasisymmetric Schur functions [21], dual immaculate quasisymmetric functions [4], type B quasisymmetric Schur functions [15,26], quasi-key polynomials [3,30] and quasisymmetric Grothendieck polynomials [25]. Their name was aptly chosen since these functions not only naturally refine Schur functions, but also generalize many classical Schur function properties, such as the Littlewood–Richardson rule from the classical [20] to the generalized [5, Theorem 3.5], the Pieri rules from the classical [27] to the generalized [13, Theorem 6.3] and the RSK algorithm from the classical [16,28,29] to the generalized [23, Procedure 3.3].

Dual to QSym is the Hopf algebra of noncommutative symmetric functions, NSym [10], whose basis dual to that of quasisymmetric Schur functions is the basis of noncommutative Schur functions [5]. By duality this basis again has a Littlewood–Richardson rule and RSK algorithm, and, due to noncommutativity, two sets of Pieri rules, one arising from multiplication on the right [32, Theorem 9.3] and one arising from multiplication on the left [5, Corollary 3.8]. Therefore in both QSym and NSym a key question in this realm remains: Are there *skew* Pieri rules for quasisymmetric and noncommutative Schur functions? In this note we give such rules that are analogous to that of their namesake Schur functions.

More precisely, the note is structured as follows. In Section 2 we review necessary notions on compositions and define operators on them. In Section 3 we recall QSym and NSym, the bases of quasisymmetric Schur functions and noncommutative Schur functions, and their respective Pieri rules. In Section 4 we give skew Pieri rules for quasisymmetric Schur functions in Theorem 4.3 and recover the Pieri rules for skew shapes of Assaf and McNamara in Corollary 4.7. We close with skew Pieri rules for noncommutative Schur functions in Theorem 4.9.

2. Compositions and diagrams

A finite list of integers $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ is called a *weak composition* if $\alpha_1, \ldots, \alpha_\ell$ are nonnegative, is called a *composition* if $\alpha_1, \ldots, \alpha_\ell$ are positive, and is called a *partition* if $\alpha_1 \geq \cdots \geq \alpha_\ell > 0$. Note that every weak composition has an underlying composition,