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Numerical polar calculus and cohomology of line bundles

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A B S T R A C T

Let $L_1, \dots, L_s$ be line bundles on a smooth complex variety $X \subset \mathbb{P}^r$ and let $D_1, \dots, D_s$ be divisors on X such that D_i represents L_i. We give a probabilistic algorithm for computing the degree of intersections of polar classes which are in turn used for computing the Euler characteristic of linear combinations of $L_1, \dots, L_s$. The input consists of generators for the homogeneous ideals $I_X, I_{D_i} \subset \mathbb{C}[x_0, \dots, x_r]$ defining X and D_i.

1. Introduction

Let $X \subset \mathbb{P}^r$ be a smooth n-dimensional variety over $\mathbb{C}$ with $n < r$. For $0 \leq j \leq n$ and $V \subseteq \mathbb{P}^r$ a general linear subspace of dimension $(r - n - 2) + j$, let

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$$P_j(X) = \{x \in X : \dim(T_x X \cap V) \geq j-1\}.$$

If X has codimension 1 and $j = 0$, then V is the empty set (rather than a linear subspace) with the convention $\dim(\emptyset) = -1$. Note that $P_j(X)$ depends on the choice of V even though we have suppressed this in the notation. However, for general V, the class $[P_j(X)]$ in the Chow ring of the variety $A_*(X)$ does not depend on V. In fact, for general V, $P_j(X)$ is either empty or of pure codimension j in X and

$$[P_j(X)] = \sum_{i=0}^{j} (-1)^i \binom{n-i+1}{j-i} H^{j-i} c_i, \tag{1}$$

where $H \in A_{n-1}(X)$ is the hyperplane class and c_i is the ith Chern class of X (see [7], Example 14.4.15). In this setting, $P_j(X)$ is called a jth *polar locus* of X and $[P_j(X)]$ is called the jth *polar class*. Because of the close relationship between polar classes and Chern classes, computation of the degrees of intersections of one is equivalent to computation of the degrees of intersections of the other. In particular the numerical algorithms developed in [1,5] for computing intersection numbers of Chern classes will be used in this note in order to develop an algorithm for computing polar degrees and the degrees of intersection of polar classes, Procedure 2 and Procedure 3. We call these computations "Numerical Polar Calculus" and will denote the algorithm by **NPC**. Immediate applications of the computation include degree of the discriminant locus and the Euclidean Distance degree, as explained in Subsection 3.1. A Macaulay2 algorithm for computing polar classes of (not necessarily smooth or normal) toric varieties has been previously developed in [11]. While applications of NPC are multiple, we focus on uses of the algorithm that we regard as particularly interesting for the applied and computational algebraic geometry community. In Proposition 4.2 and Procedure 4 we illustrate how NPC can be used to develop an algorithm for computing the *Euler Characteristic* of a linear combination of divisors on X:

$$\chi(X, \mathcal{O}(a_1D_1 + \ldots + a_sD_s)) = \sum_{i\geq 0} (-1)^i \dim(H^i(X, \mathcal{O}(a_1D_1 + \ldots + a_sD_s)))$$

where the D_i are smooth divisors on X meeting properly. More precisely, we require that any $d \leq n$ of the D_i meet in a subscheme every component of which has codimension d. The key ingredients are the Hirzebruch–Riemann–Roch formula and Adjunction formula as explained in Section 2.

We briefly illustrate the motivating idea. The Riemann–Roch theorem for a smooth projective curve X embedded in $\mathbb{P}^r$ by a line bundle L gives a powerful and striking link between invariants of the line bundle and invariants of the curve:

$$\dim(H^0(X, L)) - \dim(H^1(X, L)) = \chi(X, L) = \deg(L) + 1 - g$$