Advances in Applied Mathematics 100 (2018) 179–196

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

A context-free grammar for peaks and double descents of permutations

Amy M. Fu

School of Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, PR China

ARTICLE INFO

Article history:
Received 13 April 2018
Received in revised form 17 June 2018
Accepted 27 June 2018
Available online 17 July 2018

MSC:
05A15
05A19

Keywords:
Context-free grammars
Grammatical labeling
Exterior peaks
Proper double descents

ABSTRACT

This paper is concerned with the joint distribution of the number of exterior peaks and the number of proper double descents over permutations on $[n] = \{1, 2, \ldots, n\}$. The notion of exterior peaks of a permutation was introduced by Aguiar, Bergeron and Nyman in their study of the peak algebra. Gessel obtained the generating function of the number of permutations on $[n]$ with a given number of exterior peaks. On the other hand, by establishing differential equations, Elizalde and Noy derived the generating function for the number of permutations on $[n]$ with a given number of proper double descents. Barry and Basset deduced the generating function of the number of permutations on $[n]$ with no proper double descents. We find a context-free grammar that can be used to compute the number of permutations on $[n]$ with a given number of exterior peaks and a given number of proper double descents. Based on the grammar, the recurrence relation of the number of permutations on $[n]$ with a give number of exterior peaks can be easily obtained. Moreover, we use the grammatical calculus to derive the generating function without solving differential equations. Our formula reduces to the formulas of Gessel, Elizalde-Noy, Barry, and Basset. Finally, from the grammar we establish a relationship between our generating function and the generating function of the

E-mail address: fu.mei@mail.shufe.edu.cn.

https://doi.org/10.1016/j.aam.2018.06.004

joint distribution of the number of peaks and the number of double descents derived by Carlitz and Scoville.

1. Introduction

The objective of this paper is to present a grammatical approach to the joint distribution of exterior peaks and proper double descents of permutations on $[n]$. The notion of exterior peaks was introduced by Aguiar, Bergeron and Nyman [1]. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ on $[n]$, an index i is called an *exterior peak* if $\pi_1 > \pi_2$ for $i = 1$ or $\pi_{i-1} < \pi_i > \pi_{i+1}$ for $1 < i < n$. Let $T(n,k)$ be the number of permutations on $[n]$ with k exterior peaks and let

$$T_n(x) = \sum_{k\geq 0} T(n,k)x^k. \tag{1.1}$$

Gessel [18] obtained the generating function of $T_n(x)$.

Theorem 1.1 *(Gessel [18]). We have*

$$\sum_{n=0}^{\infty} \frac{T_n(x)t^n}{n!} = \frac{\sqrt{1-x}}{\sqrt{1-x}\cosh\left(t\sqrt{1-x}\right) - \sinh\left(t\sqrt{1-x}\right)}. \tag{1.2}$$

The number of proper double descents is a classical statistic on permutations, which has been extensively studied. An index i of a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ on $[n]$ is called a *proper double descent* if $3 \leq i \leq n$ and $\pi_{i-2} > \pi_{i-1} > \pi_i$. Denote $U(n,k)$ the number of permutations on $[n]$ with k proper double descents and let

$$U_n(y) = \sum_{k\geq 0} U(n,k)y^k. \tag{1.3}$$

By establishing the following ordinary differential equations,

$$f'' + (1-y)\left(f' + f\right) = 0$$

with $f(0) = 1$ and $f'(0) = -1$, Elizalde and Noy [10] derived the generating function of $U_n(y)$.

Theorem 1.2 *(Elizalde and Noy [10]). We have*

$$\sum_{n=0}^{\infty} \frac{U_n(y)t^n}{n!} = \frac{2\sqrt{(y-1)(y+3)}e^{t/2\cdot(1-y+\sqrt{(y-1)(y+3)})}}{1+y+\sqrt{(y-1)(y+3)} - (1+y-\sqrt{(y-1)(y+3)})e^{t\sqrt{(y-1)(y+3)}}}. \tag{1.4}$$