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Large deviations for high-dimensional random projections of ℓ_p^n-balls

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A B S T R A C T

The paper provides a description of the large deviation behavior for the Euclidean norm of projections of ℓ_p^n-balls to high-dimensional random subspaces. More precisely, for each integer $n \geq 1$, let $k_n \in \{1, \dots, n-1\}$, $E^{(n)}$ be a uniform random k_n-dimensional subspace of $\mathbb{R}^n$ and $X^{(n)}$ be a random point that is uniformly distributed in the ℓ_p^n-ball of $\mathbb{R}^n$ for some $p \in [1, \infty]$. Then the Euclidean norms $\|P_{E^{(n)}} X^{(n)}\|_2$ of the orthogonal projections are shown to satisfy a large deviation principle as the space dimension n tends to infinity. Its speed and rate function are identified, making thereby visible how they depend on p and the growth of the sequence of subspace dimensions k_n. As a key tool we prove a probabilistic representation of $\|P_{E^{(n)}} X^{(n)}\|_2$ which allows us to separate the influence of the parameter p and the subspace dimension k_n.

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1. Introduction

The geometry of convex bodies in high dimensions is a fascinating and vivid field at the core of what is known today as asymptotic geometric analysis, a branch of mathematics at the crossroads between analysis, geometry and probability theory. In particular, it has been realized in the last decades that the presence of high dimensions forces certain regularity on the geometry of convex bodies that in many instances has a probabilistic flavor, compare with the surveys of Guédon [15,16] and the monograph [7], for example. The arguably most prominent example is the central limit theorem, which is widely known in probability theory to capture the fluctuations of a sum of (independent) random variables (see, e.g., Chapter 5 in [18]). In the geometric context it roughly says that most k-dimensional marginals of a high-dimensional isotropic convex body are approximately Gaussian, provided that k is of smaller order than n^κ for some universal constant $\kappa \in (0, 1)$, i.e., $k = o(n^\kappa)$. The central limit theorem for convex bodies was conjectured in [2] by Anttila, Ball and Perissinaki (for $k = 1$), who proved the conjecture for the case of uniform distributions on convex sets whose modulus of convexity and diameter satisfy some additional quantitative assumptions. Other contributions to different facets of the central limit problem for (special classes of) convex bodies are due to Bobkov and Koldobsky [6], Brehm, Hinow, Vogt and Voigt [8], E. Meckes [24,25], E. and M. Meckes [26], E. Milman [27] or Paouris [29], just to mention a few. For general bodies, based on a principle going back to the work of Sudakov [34], and Diaconis and Freedman [11], a central limit theorem was proved by Klartag in [20,21], who obtained that $\kappa \geq 1/15$. If in addition the convex body is 1-unconditional, that is, symmetric with respect to all coordinate hyperplanes, this has been extended by M. Meckes [23] to a family of specific k-dimensional marginals with $k = o(n^{1/3})$.

On the one hand the central limit theorem underlines the universal behavior of Gaussian fluctuations. On the other hand, it is widely known in probability theory that the so-called large deviation behavior, which considers fluctuations beyond the Gaussian scale, is much more sensitive to the distributions of the involved random variables. For example, Cramér's theorem (see, e.g., [9, Theorem 2.2.3] or [18, Theorem 27.5]) guarantees that if $X, X_1, X_2, \ldots$ are independent, identically distributed and centered random variables with cumulant generating function $\Lambda(u) := \log(\mathbb{E}e^{uX}) < \infty$ for all $u \in \mathbb{R}$, one has that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_1 + \ldots + X_n \geq nt) = -\Lambda^*(t)$$

for all $t > \mathbb{E}X$, where Λ^* is the Legendre–Fenchel transform of Λ. Equivalently, this means that for any $\varepsilon > 0$ and any $t > \mathbb{E}X$ there exists some natural number n_0 so that for each $n \geq n_0$,

$$e^{-n(\Lambda^*(t)+\varepsilon)} \leq \mathbb{P}(X_1 + \ldots + X_n \geq nt) \leq e^{-n(\Lambda^*(t)-\varepsilon)}.$$