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Every binary code can be realized by convex sets

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Much work has been done to identify which binary codes can be represented by collections of open convex or closed convex sets. While not all binary codes can be realized by such sets, here we prove that every binary code can be realized by convex sets when there is no restriction on whether the sets are all open or closed. We achieve this by constructing a convex realization for an arbitrary code with k nonempty codewords in $\mathbb{R}^{k-1}$. This result justifies the usual restriction of the definition of convex neural codes to include only those that can be realized by receptive fields that are all either open convex or closed convex. We also show that the dimension of our construction cannot in general be lowered.

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1. Introduction

In neuroscience, it is common to consider collections of coactive neurons as information-coding units in neural populations. Abstractly, we encode this data as a binary code.

Definition 1. A *binary code* on n neurons is a collection of subsets $\mathcal{C}$ of the set $[n] = \{1, 2, 3, \ldots, n\}$. The elements of $\mathcal{C}$ are called *codewords*.

The primary goal of this paper is to answer the question: can every binary code be realized by convex sets, not necessarily open or closed, in $\mathbb{R}^k$? Prior to addressing the motivation for this question, we provide an example of how a binary code is extracted from a collection of convex sets.

Example 1. Consider Fig. 1 where U_1, U_2, U_3, and U_4 are convex sets in a stimulus space X.

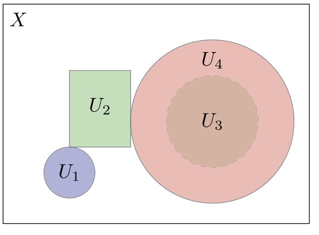

Fig. 1. Convex sets U_1, U_2, U_3, and U_4, in the stimulus space X.

The corresponding code $\mathcal{C}$ of such a collection of sets is obtained by representing each unique region cut out by U_1, U_2, U_3, and U_4 as a binary string, where the ith entry is 1 if that region is in U_i and 0 otherwise. For example, $(0, 1, 0, 0) \in \mathcal{C}$ because there exists a region of U_2 such that U_2 is not intersecting any other U_i. Moreover, $(1, 0, 0, 1) \notin \mathcal{C}$ because U_1 and U_4 are disjoint. In this way, this diagram generates the following corresponding code: $\mathcal{C} = \{(0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 1), (1, 1, 0, 0), (0, 1, 0, 1), (0, 0, 1, 1)\}$. We say that $\mathcal{C}$ is the binary code arising from our diagram. In this paper, we will write our code using the supports of the binary strings: $\mathcal{C} = \{\emptyset, 1, 2, 4, 12, 24, 34\}$. Since U_1, U_2, U_3, and U_4 are convex, we say that $\mathcal{C}$ is convex realizable. The focus of this paper will be on determining when a neural code is convex realizable.

Our study of binary codes is motivated by biological research into a type of neuron called a place cell. In 1971, John O'Keefe discovered place cells, an accomplishment for which he shared the 2014 Nobel Prize in Physiology or Medicine. A place cell is a neuron that fires when an animal is in a particular location relative to its environment and thus provides an internal representation of the animal's spatial location. These particular locations, called receptive fields, are approximately convex. From the regions cut out by the receptive fields, we obtain a binary code called a neural code [4].