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On factor-free Dyck words with half-integer slope

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ABSTRACT

We study a class of rational Dyck paths with slope $\frac{2m+1}{2}$ corresponding to factor-free Dyck words, as introduced by P. Duchon. We show that, for the slopes considered in this paper, the language of factor-free Dyck words is generated by an auxiliary language that we examine from the algebraic and combinatorial points of view. We provide a lattice path description of this language, and give an explicit enumeration formula in terms of partial Bell polynomials. As a corollary, we obtain new formulas for the number of associated factor-free generalized Dyck words.

1. Introduction

In these notes we consider the set of factor-free words belonging to the generalized Dyck language constructed from the alphabet $\mathcal{A} = \{a, b\}$ with valuations $h(a) = 2m+1$, $m \in \mathbb{N}$, and $h(b) = -2$. This is an instance of the two-letter Dyck language studied by P. Duchon for which words correspond to Dyck paths with rational slope. In the case at hand, the slope is $\frac{2m+1}{2}$.

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In [9], Duchon provided an algebraic grammar for generalized Dyck languages (as introduced by Labelle and Yeh [10]) and proved that words in such a language can be obtained uniquely by inserting words of the language into factor-free words of the same language.

Recall that a *factor* of a word w is any word w' such that $w = w_1 w' w_2$. If w_1 and w_2 are not both the empty word, w' is a *proper factor* of w. For a given alphabet $\mathcal{A}$ we let $\mathcal{A}^*$ denote the set of all words made from $\mathcal{A}$, together with the empty word ε. A word w in $\mathcal{A}^*$ is said to be a *generalized Dyck word* if it satisfies the conditions that $h(w) = 0$, and for each left factor w_1 of w, $h(w_1) \geq 0$. We denote by $\mathscr{D}_{\mathcal{A},h}$ the set of generalized Dyck words over the alphabet $\mathcal{A}$ with valuation given by h. Moreover, we say that a word in $\mathcal{A}^*$ is $\mathscr{D}_{\mathcal{A},h}$-factor-free (or simply factor-free if the underlying Dyck language is clear) if it has no proper factor belonging to $\mathscr{D}_{\mathcal{A},h}$. The set of factor-free words in $\mathscr{D}_{\mathcal{A},h}$ will be denoted by $D_{\mathcal{A},h}$.

As shown in [9, Section 5], the algebraic grammars for $\mathscr{D}_{\mathcal{A},h}$ and $D_{\mathcal{A},h}$ can be described by a system of derivation rules in terms of certain auxiliary languages with restrictions on their total and partial valuations. As we will see in Section 2, for Dyck words with slope $\frac{2m+1}{2}$, the aforementioned derivation rules may be reduced to a single core language that we denote by U, or $U_{\frac{2m+1}{2}}$ if we wish to emphasize the slope.

The main focus of this paper is to study the auxiliary language U from the algebraic and combinatorial points of view. We provide a description of U in terms of lattice paths and, based on a polynomial equation satisfied by the generating function, we give an explicit enumeration formula involving partial Bell polynomials. As a corollary, we obtain new formulas for the enumeration of the corresponding factor-free generalized Dyck words. In Section 4 we illustrate our results for slopes $\frac{3}{2}$ and $\frac{5}{2}$. In particular, we discuss a bijection between the elements of $U_{\frac{5}{2}}$ and certain colored trees having nonleaf nodes of outdegrees 2 or 4. In the last section of the paper, we provide the building blocks needed to create factor-free words with slope $\frac{7}{2}$, we give an interesting connection between the auxiliary language U and certain colored Dyck paths, and we briefly discuss the use of factor-free Dyck words to generate cross-bifix-free (non-overlapping) codes of binary words with variable length.

2. Grammar for factor-free words and the auxiliary language U

In this section, we will review some of the terminology introduced by Duchon in [9] and will discuss the reduced algebraic grammar for the two-letter sublanguage of factor-free Dyck words with slope $\frac{2m+1}{2}$. In addition, we define the language U alluded to in the introduction and describe its elements in terms of lattice paths.

First, consider the auxiliary languages L_i and R_j defined as follows:

- L_i is the set of factor-free words $w \in \mathcal{A}^*$ with total valuation i, such that each nonempty left factor w_1 of w has $h(w_1) > i$,