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A-hypergeometric distributions and Newton polytopes

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A R T I C L E I N F O

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A B S T R A C T

We give a bijection between a quotient space of the parameters and the space of moments for any A-hypergeometric distribution. An algorithmic method to compute the inverse image of the map is proposed utilizing the holonomic gradient method and an asymptotic equivalence of the map and the iterative proportional scaling. The algorithm gives a method for solving a conditional maximum likelihood estimation problem in statistics. The interplay between the theory of hypergeometric functions and statistics allows us to give some new formulas for A-hypergeometric polynomials.

1. Introduction

We denote by $\mathbb{N}$ the set of the non-negative integers. Let A be a $d \times n$ configuration matrix with non-negative integer entries. We assume that the rank of A is d. The A-hypergeometric polynomial [23] for A and $\beta \in \mathbb{N}^d$ is defined by

$$Z(\beta; p) = \sum_{Au=\beta, u \in \mathbb{N}^n} \frac{p^u}{u!}, \qquad (1)$$

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where $p^u = \prod_{i=1}^n p_i^{u_i}$ and $u! = \prod_{i=1}^n u_i!$. Set $p_i = \exp \xi_i$ and let $\exp \xi$ denote the vector $(\exp \xi_1, \ldots, \exp \xi_n)$. We fix $\beta \neq 0$ such that $\beta \in \mathbb{N}A = \sum_{i=1}^n \mathbb{N}a_i$, where a_i denotes the i-th column vector of the matrix A. Let $U \in \mathbb{N}^n$ be a random variable of the (A, β) *hypergeometric distribution with the parameter* $p \in \mathbb{R}^n_{>0}$ (or $\xi \in \mathbb{R}^n$), which is defined by

$$P(U = u \,|\, Au = \beta) = \frac{p(\xi)^u}{u! Z(\beta; p(\xi))} = \frac{\exp(u \cdot \xi)}{u! Z(\beta; p(\xi))}, \quad u \cdot \xi = \sum_{i=1}^n u_i \xi_i. \tag{2}$$

If no confusion arises, we simply call this the A-hypergeometric distribution. The A-hypergeometric distribution is in turn a generalization of the generalized (p_i may take any positive number) hypergeometric distribution on the contingency tables with fixed marginal sums (see, e.g., [11, Chapters 4, 6], [17]), and is the conditional distribution of u given by $\beta = Au$ under the Poisson distribution

$$P(U = u) = \frac{p^u}{u!} \exp(-\mathbf{1} \cdot p), \quad \mathbf{1} = (1, \ldots, 1). \tag{3}$$

In the setting of testing statistical hypotheses, this corresponds to the alternative hypothesis against the null hypothesis of $p_i = s^{a_i}$, $s \in \mathbb{R}^d_{>0}$, $\forall i$. The polynomial Z is the *normalizing constant* or the *partition function* of the A-hypergeometric distribution.

The expectation of the random variable U_i under (2) is equal to

$$\sum_{Au=\beta, u \in \mathbb{N}^n} u_i \frac{p(\xi)^u}{u! Z(\beta; p(\xi))}. \tag{4}$$

Setting

$$\psi(\xi) = \log Z(\beta; p(\xi)),$$

the expectation of U_i is written as

$$E[U_i] = \frac{p_i \partial_i \bullet Z}{Z}|_{p=p(\xi)} = \frac{\partial \psi(\xi)}{\partial \xi_i},$$

where $p = (\exp \xi_1, \ldots, \exp \xi_n)$, $\partial_i = \frac{\partial}{\partial p_i}$ and $\partial_i \bullet Z = \frac{\partial Z}{\partial p_i}$. If we set $\eta_i = E[U_i]$ and $\eta = (\eta_i)$, which is a function of ξ, then the ξ-space and the η-space are dual by the *moment map* $E[U]$ in the context of the information geometry [1].

We study here the map between the ξ-space (the space of the parameters) and the η-space (the space of moments). This correspondence has been studied from several points of view in statistics and information geometry, e.g., [1], [2], [3], [8], [10]. In Section 2, we determine the image of the ξ-space $\mathbb{R}^n$ by the moment map in the η-space, which is described in terms of the Newton polytope of the polynomial Z. We introduce a quotient space, which is called the space of *the generalized odds ratios*, of the ξ-space and construct an isomorphism between the quotient space and the Newton polytope in