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Computation of all rational solutions of first-order algebraic ODEs

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ABSTRACT

In this paper, we discuss three different approaches to attack the problem of determining all rational solutions for a first-order algebraic ordinary differential equation (AODE). We first give a sufficient condition for first-order AODEs to have the property that poles of rational solutions can only occur at the zeros of the leading coefficient. A combinatorial approach is presented to determine all rational solutions, if there are any, of the family of first-order AODEs satisfying this condition. Algebraic considerations based on algebraic function theory yield another algorithm for quasi-linear first-order AODEs. And finally ideas from algebraic geometry combine these results to an algorithm for finding all rational solutions of parametrizable first-order AODEs.

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1. Introduction

A first-order AODE is a differential equation of the form $F(x, y, \frac{dy}{dx}) = 0$ for some trivariate polynomial F with coefficients in an algebraically closed field $\mathbb{K}$ of characteristic zero. We assume F to be irreducible. The study of first-order AODEs has a long history which can be traced back to the works by Fuchs [6] and Poincaré [16]. Malmquist [12] provided a necessary criterion for a first-order AODE to have a transcendental meromorphic solution. In the second half of the twentieth century, the development of differential algebra provided powerful tools for studying algebraic differential equations. Matsuda [13] classified differential function fields having no movable singularity up to isomophism of differential fields. As a nice application of the theory by Matsuda, Eremenko [3] presented a theoretical consideration on a degree bound of rational solutions for first-order AODEs.

There is a variety of solution methods for special classes of first-order AODEs, though no general algorithm exists so far. Kovacic [10] gives an algorithm for finding all algebraic solutions of a rational Riccati equation, and is then able to find all Liouvillian solutions of a second-order linear ODEs as a by-product. Carnicer [1] studied a degree bound for algebraic function solutions for first-order first-degree AODEs. As an application of Gröbner bases, Hubert [8] suggested an algorithm for implicitly determining solutions of a first-order AODE. An algebraic geometric method for solving such AODEs can be found in [4,14,15,7,20].

The problem of determining a closed form solution for a first-order AODE is usually a challenge. Even if a closed form solution cannot be achieved in general, it is still interesting to ask for the existence of special kinds of solutions. In this paper, we are interested in rational solutions, i.e. solutions which are rational functions in x.

The problem of computing all rational solutions for a first-order AODE has received much attention in the last decade. Feng and Gao [4,5] proposed an algorithm for computing a rational general solution for an autonomous first-order AODE. Their algorithm runs in polynomial time. Ngô and Winkler [14,15] generalized the idea by Feng and Gao to the class of non-autonomous first-order AODEs. A similar idea can be found in [2]. However, the algorithms proposed by Ngô and Winkler [14,15], and by Chen and Ma [2] were not complete. By a slight modification of the algorithm for determining an optimal parametrization of an algebraic curve over a rational function field, Vo et al. [19] overcame the missing steps and obtained a decision algorithm.

A first-order AODE having no rational general solution may still admit several rational solutions. These other rational solutions cannot be detected by the algorithms developed in [4,14,2,19]. Therefore, the problem of determining all rational solutions for a first-order AODE is still interesting. As far as we are aware, even for the class of first-order first-degree AODEs, no general algorithm for deciding the existence of a rational solution and, in the affirmative case, computing all of them exists. In this paper, we discuss three approaches to attack the problem. The scope of the obtained algorithms covers a big part of first-order AODEs. In particular, it covers the class of parametrizable