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Phase transitions in stochastic non-linear threshold Boolean automata networks on $\mathbb{Z}^2$: The boundary impact

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A B S T R A C T

This paper addresses the question of the impact of the boundary on the dynamical behaviour of finite Boolean automata networks on Z2. The evolution over discrete time of such networks is governed by a specific stochastic threshold non-linear transition rule derived from the classical rule of formal neural networks. More precisely, the networks considered in this paper are finite but the study is done for arbitrarily large sizes. Moreover, the boundary impact is viewed as a classical definition of a phase transition in probability theory, characterising in our context the fact that a network admits distinct asymptotic behaviours when different boundary instances are assumed. The main contribution of this paper is the highlight of a formula for a necessary condition for boundary sensitivity, whose sufficiency and necessity are entirely proven with natural constraints on interaction potentials.

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1. Introduction

Understanding the influence of the frontiers (or boundaries, or environment) of systems composed of interacting entities is a problem born in the 1920's. In physics, after the seminal paper of Ising in 1925 dealing with phase transitions in ferromagnetic systems [1] and the first theoretical proof of their existence by Onsager in 1944 [2], the most important works on this subject are certainly those that focused at the end of the 1960's on lattice gas models. Amongst these works, those of Dobrushin [3,4] and Ruelle [5,6] are obviously the most known in the sense that they presented the first results proving that the Ising model embedded into a square lattice admits a phase transition depending on the nature of its boundary conditions. Even if these works were dived in physics, they opened many questions in other disciplines. Indeed, issues underlying the role of boundaries on systems is all the more pertinent in frameworks at the frontier of theoretical computer science and biology. For instance, boundary conditions may allow to represent the post-transcriptional actions of non-coding RNAs in the genetic context [7], external electric fields in the neural context [8], and also hormone flows control in both of these [9].

Since decades, researches in discrete mathematics and fundamental computer science have put the emphasis on the modelling abilities of automata networks concerning interaction networks. In particular, since their introduction in the works of McCulloch and Pitts [10] and Kauffman [11,12], Boolean automata networks (BANs for short) have been at the centre of numerous studies in the field of biological networks modelling, like neural networks [13–18], genetic regulation networks [19–25] and more recently social networks [26,27]. This can be easily explained by their very high level of abstraction that makes them ideal objects to capture formally the essence of interactions and to focus on qualitative aspects of their dynamics (*e.g.*, the information transmissions).

In this paper, our attention has focused on a fundamental analysis of the asymptotic dynamical behaviours of a particular class of BANs on $\mathbb{Z}^2$ subjected to the influence of distinct boundary instances. Previous works on *linear stochastic threshold* BAN*s* (LSBANs for short) showed that the sensitivity of such BANs against their boundary is quite similar to that of the Ising model [28,29]. In the same lines and on the basis of preliminary results [30], the main contribution of this paper is an explicit formula of a necessary condition according to which *non-linear stochastic threshold* BAN*s* (NSBANs for short) are effectively subjected to the impact of changes of their boundary instances. Our interest in non-linearity comes from the fact that non-linearity is an original way to model entity coalitions. For instance, about biological regulation networks, it gives a way to represent protein complexes [23,31,32] inside the local transition functions, which prevents from transforming the structural features of networks by adding vertices and edges to their underlying interaction graphs. Thus, non-linearity constitutes a means to explicit cooperative or competitive coalitions without increasing problem sizes (*i.e.*, the sizes of their inputs).