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Bumping sequences and multispecies juggling ☆

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Building on previous work by four of us (ABCN), we consider further generalizations of Warrington's juggling Markov chains. We first introduce "multispecies" juggling, which consist in having balls of different weights: when a ball is thrown it can possibly bump into a lighter ball that is then sent to a higher position, where it can in turn bump an even lighter ball, etc. We both study the case where the number of balls of each species is conserved and the case where the juggler sends back a ball of the species of its choice. In this latter case, we actually discuss three models: add-drop, annihilation and overwriting. The first two are generalisations of models presented in (ABCN) while the third one is new and its Markov chain has the ultra fast convergence property. We

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finally consider the case of several jugglers exchanging balls. In all models, we give explicit product formulas for the stationary probability and closed form expressions for the normalisation factor if known.

1. Introduction

Several Markov chains studied in nonequilibrium statistical physics are known to have, despite nontrivial dynamics, an explicit and sometimes remarkably simple stationary state. The most famous examples of these are one-dimensional models of hopping particles such as the asymmetric exclusion process [5], where the stationary state satisfies the so-called matrix product representation [4] and the zero-range process, where the stationary state is factorised [8]. The main reason for this simplicity is the underlying combinatorial structure of these processes. Of the two examples mentioned above, a variant of the former known as the totally asymmetric simple exclusion process (TASEP), solved first in [6], has a rich combinatorial structure even when the system is generalised to include several types of particles. The latter system is known as the *multispecies TASEP*, and its stationary state has an explicit solution which comes from queueing theory [9].

The multispecies TASEP has the further exceptional property that the stationary state can also be calculated if the hopping probabilities of particles depend on their location, known as the *inhomogeneous multispecies TASEP*. This was first done for the three-species case in [3] and the result for arbitrary species has been announced in [12]. While the stationary state of the general inhomogeneous multispecies TASEP has an explicit description in principle, the actual formulas for the stationary probabilities can be considerably complicated.

In this paper, we will first study the multispecies variants of the basic juggling process introduced in [13] then extended to their inhomogeneous versions in [7,1]. In contrast to the TASEP, as we will show in Theorem 6, the stationary probabilities and the partition function have elegant and compact expressions. We then study the multispecies variants of two other juggling processes, which were also introduced in [13], where the number of balls of each type can vary. In all of these cases, we prove analogous results; see Theorems 10 and 12. We also introduce a new model where the number of balls of each type can vary that we call the overwriting model. This model has the nice property that it converges to its stationary distribution in deterministic finite time. In probabilistic language, this is equivalent to saying that the overwriting model has a deterministic strong stationary time.

The rest of the paper is organized as follows. In Section 2, we discuss in some detail the first model, the so-called Multispecies Juggling Markov Chain (MSJMC): Section 2.1 provides its definition and the expression for its stationary distribution, and Section 2.2