Advances in Applied Mathematics 97 (2018) 27–35

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Apéry sets of shifted numerical monoids

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ARTICLE INFO

Article history:
Received 2 September 2017
Received in revised form 29 January 2018
Accepted 29 January 2018
Available online xxxx

MSC:
primary 20M14, 05E40

Keywords:
Numerical semigroup
Computation
Quasipolynomial

ABSTRACT

A numerical monoid is an additive submonoid of the non-negative integers. Given a numerical monoid S, consider the family of "shifted" monoids M_n obtained by adding n to each generator of S. In this paper, we characterize the Apéry set of M_n in terms of the Apéry set of the base monoid S when n is sufficiently large. We give a highly efficient algorithm for computing the Apéry set of M_n in this case, and prove that several numerical monoid invariants, such as the genus and Frobenius number, are eventually quasipolynomial as a function of n.

1. Introduction

The factorization theory of numerical monoids – co-finite, additive submonoids of the non-negative integers – has enjoyed much recent attention; in particular, invariants such as the minimum factorization length, delta set, and ω-primality have been studied in much detail [10]. These measures of non-unique factorization for individual elements in a numerical monoid all exhibit a common feature: eventual quasipolynomial behavior.

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https://doi.org/10.1016/j.aam.2018.01.005

In many cases, this eventual behavior is periodic (i.e. quasiconstant) or quasilinear, and this pattern always holds after some initial "noise" for small monoid elements.

In this paper, we describe quasipolynomial behavior of certain numerical monoid invariants over parameterized monoid families. Unlike previous papers, which studied how factorization invariants change element-by-element (e.g., minimum factorization length [1]), we investigate how a monoid's properties change as the generators vary by a shift parameter. More specifically, we study "shifted" numerical monoids

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle$$

for $r_1 < \cdots < r_k$, and find explicit relationships between the Frobenius number, genus, type, and other properties of M_n and M_{n+r_k} when $n > r_k^2$. As with the previous element-wise investigations of invariant values, our monoid-wise analysis reveals eventual quasipolynomial behavior, this time with respect to the shift parameter n.

The main result of this paper is Theorem 3.3, which characterizes the Apéry set of M_n (Definition 2.2) for large n in terms of the Apéry set of the monoid $S = \langle r_1, \ldots, r_k \rangle$ at the base of the shifted family. Apéry sets are non-minimal generating sets that concisely encapsulate much of the underlying monoid structure, and many properties of interest can be recovered directly and efficiently from the Apéry set, making it a sort of "one stop shop" for computation. We utilize these connections in Section 4 to derive relationships between properties of M_n and M_{n+r_k} when n is sufficiently large.

One of the main consequences of our results pertains to computation. Under our definition of M_n above, every numerical monoid is a member of some shifted family of numerical monoids. While Apéry sets of numerical monoids (and many of the properties derived from them) are generally more difficult to compute when the minimal generators are large, our results give a way to more efficiently perform these computations by instead computing them for the numerical monoid S, which has both smaller and fewer generators than M_n. In fact, one surprising artifact of the algorithm described in Remark 3.5 is that, in a shifted family $\{M_n\}$ of numerical monoids, the computation of the Apéry set of M_n for $n > r_k^2$ is typically significantly faster than for M_n with $n \le r_k^2$, even though the former has larger generators. We discuss this and further computational consequences in Remark 4.10, including implementation of our algorithm in the popular `GAP` package `numericalsgps` [4].

2. Background

In this section, we recall several definitions and results used in this paper. For more background on numerical monoids, we direct the reader to [11].

Definition 2.1. A *numerical monoid* M is an additive submonoid of $\mathbb{Z}_{\ge 0}$. Whenever we write $M = \langle n_1, \ldots, n_k \rangle$, we assume that $n_1 < \cdots < n_k$. We say M is *primitive* if $\gcd(n_1, \ldots, n_k) = 1$. A *factorization* of an element $a \in M$ is an expression