Advances in Applied Mathematics 97 (2018) 36–53

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

On the Andrews–Warnaar identities for partial theta functions

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ARTICLE INFO

Article history:
Received 18 June 2017
Received in revised form 19 February 2018
Accepted 20 February 2018
Available online xxxx

MSC:
primary 05A30
secondary 33D15

Keywords:
Ramanujan
Partial theta function
Partial theta function identity
Jacobi's triple product
Summation
Transformation
Bailey pair
Bivariate representation

ABSTRACT

In this paper we set up a bivariate representation of partial theta functions which not only unifies some famous identities for partial theta functions due to Andrews and Warnaar but also unveils a new characteristic of such identities. As further applications, we establish a general form of Warnaar's identity and a general q-series transformation associated with Bailey pairs via the use of the power series expansion of partial theta functions.

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[1] This work was supported by NSFC grant No. 11471237.

https://doi.org/10.1016/j.aam.2018.02.002
0196-8858/

1. Introduction

Throughout this paper, we adopt the standard notation and terminology for q-series from the book [15] (Gasper and Rahman, 2004). As customary, the q-shifted factorials of complex variable x with the base q are given by

$$(x;q)_\infty := \prod_{n=0}^{\infty}(1-xq^n) \qquad \text{and}$$

$$(x;q)_n := \frac{(x;q)_\infty}{(xq^n;q)_\infty}$$

for all integers n. For integers $m \geq 1$, we employ the multi-parameter compact notation

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \ldots (a_m;q)_\infty.$$

In [15], Heine's ${}_2\phi_1$ basic hypergeometric series with the base q and the argument x is defined as

$${}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, x\right] := \sum_{n=0}^{\infty} \frac{(a,b;q)_n}{(q,c;q)_n} x^n.$$

Sums of the form

$$\sum_{n=0}^{\infty} q^{An^2+Bn} x^n \quad (A > 0) \tag{1.1}$$

are called partial theta functions owing to the fact that

$$\sum_{n=-\infty}^{\infty} q^{An^2+Bn} x^n \quad (x \neq 0, A > 0)$$

is often referred to as the (complete) theta function. For the complete theta function, there holds the famous Jacobi triple product identity [15, (II.28)]

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} x^n = (q, x, q/x; q)_\infty. \tag{1.2}$$

Following [4,5], any q-series identity containing sums such as (1.1) is called a partial theta function identity. Partial theta function identities first appeared in Ramanujan's legendary lost notebook [36], wherein he recorded many identities of such sort without any proofs. A full treaty together with a more complete bibliography on this topic can be found in [7, Chap. 6] by G.E. Andrews and B.C. Berndt. Here, we choose from [7] some typical identities as illustration.