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Markov random fields and iterated toric fibre products

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We prove that iterated toric fibre products from a finite collection of toric varieties are defined by binomials of uniformly bounded degree. This implies that Markov random fields built up from a finite collection of finite graphs have uniformly bounded Markov degree.

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1. Introduction and main results

The notion of *toric fibre product* of two toric varieties goes back to [18]. It is of relevance in algebraic statistics since it captures algebraically the Markov random field on a graph obtained by gluing two graphs along a common subgraph; see [13] and also below. In [18,13,12] it is proved that under certain conditions, one can explicitly construct a Markov basis for the large Markov random field from bases for the components. For related results see [16,5,8].

However, these conditions are not always satisfied. Nevertheless, in [13, Conjecture 56] the hope was raised that when building larger graphs by gluing copies from a finite collection of graphs along a common subgraph, there might be a uniform upper bound on the Markov degree of the models thus constructed, independent of how many copies of each graph are used. A special case of this conjecture was proved in the same paper [13, Theorem 54]. We prove the conjecture in general, and along the way we link it to recent work [17] in *representation stability*. Indeed, an important point we would like to make, apart from proving said conjecture, is that algebraic statistics is a natural source of problems in *asymptotic algebra*, to which ideas from representation stability apply. Our main theorems are reminiscent of Sam's recent stabilisation theorems on equations and higher syzygies for secant varieties of Veronese embeddings [14,15].

1.1. Markov random fields

Let $G = (N, E)$ be a finite, undirected, simple graph and for each node $j \in N$ let X_j be a random variable taking values in the finite set $[d_j] := \{1, \dots, d_j\}$. A joint probability distribution on $(X_j)_{j\in N}$ is said to satisfy the *local Markov properties* imposed by the graph if for any two non-neighbours $j, k \in N$ the variables X_j and X_k are conditionally independent given $\{X_l \mid \{j, l\} \in E\}$.

On the other hand, a joint probability distribution f on the X_j is said to *factorise according to* G if for each maximal clique C of G and configuration $\alpha \in \prod_{j\in C}[d_j]$ of the random variables labelled by C there exists an interaction parameter θ^C_α such that for each configuration $\beta \in \prod_{j\in N}[d_j]$ of all random variables of G:

$$f(\beta) = \prod_{C\in \mathrm{mcl}(G)} \theta^C_{\beta|_C}$$

where $\mathrm{mcl}(G)$ is the set of maximal cliques of G, and $\beta|_C$ is the restriction of β to C.

These two notions are connected by the Hammersley–Clifford theorem, which says that a positive joint probability distribution on G factorises according to G if and only if it satisfies the Markov properties; see [6] or [9, Theorem 3.9].