

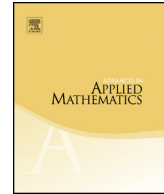


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# Markov random fields and iterated toric fibre products



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## ABSTRACT

We prove that iterated toric fibre products from a finite collection of toric varieties are defined by binomials of uniformly bounded degree. This implies that Markov random fields built up from a finite collection of finite graphs have uniformly bounded Markov degree.

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## 1. Introduction and main results

The notion of *toric fibre product* of two toric varieties goes back to [18]. It is of relevance in algebraic statistics since it captures algebraically the Markov random field on a graph obtained by gluing two graphs along a common subgraph; see [13] and also below. In [18,13,12] it is proved that under certain conditions, one can explicitly construct a Markov basis for the large Markov random field from bases for the components. For related results see [16,5,8].

However, these conditions are not always satisfied. Nevertheless, in [13, Conjecture 56] the hope was raised that when building larger graphs by gluing copies from a finite collection of graphs along a common subgraph, there might be a uniform upper bound on the Markov degree of the models thus constructed, independent of how many copies of each graph are used. A special case of this conjecture was proved in the same paper [13, Theorem 54]. We prove the conjecture in general, and along the way we link it to recent work [17] in *representation stability*. Indeed, an important point we would like to make, apart from proving said conjecture, is that algebraic statistics is a natural source of problems in *asymptotic algebra*, to which ideas from representation stability apply. Our main theorems are reminiscent of Sam’s recent stabilisation theorems on equations and higher syzygies for secant varieties of Veronese embeddings [14,15].

### 1.1. Markov random fields

Let  $G = (N, E)$  be a finite, undirected, simple graph and for each node  $j \in N$  let  $X_j$  be a random variable taking values in the finite set  $[d_j] := \{1, \dots, d_j\}$ . A joint probability distribution on  $(X_j)_{j \in N}$  is said to satisfy the *local Markov properties* imposed by the graph if for any two non-neighbours  $j, k \in N$  the variables  $X_j$  and  $X_k$  are conditionally independent given  $\{X_l \mid \{j, l\} \in E\}$ .

On the other hand, a joint probability distribution  $f$  on the  $X_j$  is said to *factorise according to  $G$*  if for each maximal clique  $C$  of  $G$  and configuration  $\alpha \in \prod_{j \in C} [d_j]$  of the random variables labelled by  $C$  there exists an interaction parameter  $\theta_\alpha^C$  such that for each configuration  $\beta \in \prod_{j \in N} [d_j]$  of all random variables of  $G$ :

$$f(\beta) = \prod_{C \in \text{mcl}(G)} \theta_{\beta|_C}^C$$

where  $\text{mcl}(G)$  is the set of maximal cliques of  $G$ , and  $\beta|_C$  is the restriction of  $\beta$  to  $C$ .

These two notions are connected by the Hammersley–Clifford theorem, which says that a positive joint probability distribution on  $G$  factorises according to  $G$  if and only if it satisfies the Markov properties; see [6] or [9, Theorem 3.9].

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