Advances in Applied Mathematics 96 (2018) 18–38

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Some useful theorems for asymptotic formulas and their applications to skew plane partitions and cylindric partitions

Guo-Niu Han, Huan Xiong *

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France

ARTICLE INFO

Article history:
Received 18 July 2017
Received in revised form 18 December 2017
Accepted 19 December 2017
Available online xxxx

MSC:
05A16
05A17

Keywords:
Asymptotic formula
Integer partition
Plane partition
Cylindric partition

ABSTRACT

Inspired by the works of Dewar, Murty and Kotěšovec, we establish some useful theorems for asymptotic formulas. As an application, we obtain asymptotic formulas for the numbers of skew plane partitions and cylindric partitions. We prove that the order of the asymptotic formula for the number of skew plane partitions of fixed width depends only on the width of the region, not on the profile (the skew zone) itself, while this is not true for cylindric partitions.

1. Introduction

Inspired by the works of Dewar, Murty and Kotěšovec [7,12], we establish some useful theorems for asymptotic formulas. Define

* Corresponding author.
E-mail addresses: guoniu.han@unistra.fr (G.-N. Han), xiong@math.unistra.fr (H. Xiong).

https://doi.org/10.1016/j.aam.2017.12.007

$$\psi_n(v,r,b;p) := v\sqrt{\frac{p(1-p)}{2\pi}}\frac{r^{b+(1-p)/2}}{n^{b+1-p/2}}\exp(n^p r^{1-p}) \tag{1.1}$$

for $n \in \mathbb{N}$, $v, b \in \mathbb{R}$, $r > 0$, $0 < p < 1$. The following is our main result.

Theorem 1.1. *Let t_1 and t_2 be two given positive integers with $gcd(t_1, t_2) = 1$. Suppose that*

$$F_1(q) = \sum_{n=0}^{\infty} a_{t_1 n} q^{t_1 n} \qquad \text{and} \qquad F_2(q) = \sum_{n=0}^{\infty} c_{t_2 n} q^{t_2 n}$$

are two power series such that their coefficients satisfy the asymptotic formulas

$$a_{t_1 n} \sim t_1 \psi_{t_1 n}(v_1, r_1, b_1; p), \tag{1.2}$$

$$c_{t_2 n} \sim t_2 \psi_{t_2 n}(v_2, r_2, b_2; p), \tag{1.3}$$

where $v_1, b_1, v_2, b_2 \in \mathbb{R}$, $r_1, r_2 > 0$, $0 < p < 1$. Then, the coefficients d_n in the product

$$F_1(q)F_2(q) = \sum_{n=0}^{\infty} d_n q^n$$

satisfy the following asymptotic formula

$$d_n \sim \psi_n(v_1 v_2, r_1 + r_2, b_1 + b_2; p). \tag{1.4}$$

Some special cases of Theorem 1.1 have been established. In 2013, Dewar and Murty [7] proved the case of $p = 1/2$, $t_1 = t_2 = 1$. Later, Kotěšovec [12] obtained the case of $0 < p < 1$, $t_1 = t_2 = 1$. We add two more parameters t_1 and t_2 in order to calculate the asymptotic formulas for plane partitions, without them Theorem 1.2 would not be proven. Our further contribution is to reformulate the asymptotic formula in a much simpler form (1.2), (1.3) and (1.4), so that the result of Theorem 1.1 can be easily iterated for handling a product of multiple power series $F_1(q)F_2(q)\cdots F_k(q)$ (see Theorem 2.3). We also obtain the following two theorems, which are useful to find asymptotic formulas for various plane partitions.

Theorem 1.2. *Let m be a positive integer. Suppose that x_i and y_i $(1 \le i \le m)$ are positive integers such that $\gcd(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m) = 1$. Then, the coefficients d_n in the following infinite product*

$$\prod_{i=1}^{m}\prod_{k\ge 0}\frac{1}{1-q^{x_i k + y_i}} = \sum_{n=0}^{\infty} d_n q^n$$

have the following asymptotic formula