Advances in Applied Mathematics 96 (2018) 76–138

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

# Minimal length maximal green sequences

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A R T I C L E   I N F O

*Article history:*
Received 16 March 2017
Received in revised form 7 December 2017
Accepted 12 December 2017

*MSC:*
05E99
05E10
16G20
13F60

*Keywords:*
Quiver mutation
Triangulated surface
Scattering diagram
Maximal green sequence

A B S T R A C T

Maximal green sequences are important objects in representation theory, cluster algebras, and string theory. It is an open problem to determine what lengths are achieved by the maximal green sequences of a quiver. We combine the combinatorics of surface triangulations and the basics of scattering diagrams to address this problem. Our main result is a formula for the length of minimal length maximal green sequences of quivers defined by triangulations of an annulus or a punctured disk.



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https://doi.org/10.1016/j.aam.2017.12.008
0196-8858/

## 1. Introduction

A maximal green sequence is a distinguished sequence of local transformations, known as **mutations**, of a given **quiver** (i.e., directed graph). Maximal green sequences were introduced by Keller in [18] in order to obtain combinatorial formulas for the refined Donaldson–Thomas invariants of Kontsevich and Soibelman [19]. They are also important in string theory [2], representation theory [5,7], and cluster algebras [14].

Recently, there have been many developments on the combinatorics of maximal green sequences (see [21] and references therein). In particular, in [21] it is shown that almost any quiver $Q$ with a finite **mutation class** (i.e., the set of quivers obtained from $Q$ by mutations) has a maximal green sequence. Our goal is to add to the known combinatorics by developing a numerical invariant of the set of maximal green sequences of $Q$: the length of minimal length maximal green sequences.

This invariant is natural from the perspective of cluster algebras. Fixing a quiver $Q$ induces an orientation of the edges of the corresponding exchange graph. It turns out that the maximal green sequences of $Q$ are in natural bijection with finite length maximal directed paths of the resulting oriented exchange graph (see [5]). Examples of oriented exchange graphs include the Hasse diagrams of Tamari lattices and of Cambrian lattices of type $\mathbb{A}$, $\mathbb{D}$, and $\mathbb{E}$ [24]. In these examples, the minimal length maximal green sequences always have length equal to the number of vertices of $Q$, but, in general, the minimal length of a maximal green sequence may be larger than the number of vertices of $Q$. Thus understanding the length of minimal length maximal green sequences provides new information about oriented exchange graphs.

Our main results (see Theorems 6.1 and 9.2) are formulas for this minimal length when $Q$ is of **mutation type** $\mathbb{D}_n$ or **mutation type** $\widetilde{\mathbb{A}}_n$ (i.e., $Q$ is in the mutation class of a type $\mathbb{D}_n$ or of an affine type $\mathbb{A}_n$ quiver). This number was calculated in mutation type $\mathbb{A}$ in [10]. We also obtain explicit constructions of minimal length maximal green sequences of $Q$, in the process of proving Theorems 6.1 and 9.2.

From a quiver $Q$ (with potential), one may construct a 3-dimensional Calabi–Yau variety, which has an associated Donaldson–Thomas (DT) invariant. DT-invariants may be defined geometrically or combinatorially. In the combinatorial setting, one translates a maximal green sequence as a factorization of a DT-invariant into a product of quantum dilogarithms [18]. So, for the purpose of calculating DT-invariants, it is useful to have a shortest factorization, i.e. a minimal length maximal green sequence.

We also suspect that the length of minimal length maximal green sequences may be significant from the perspective of representation theory of algebras. More specifically, given two quivers that define derived equivalent cluster-tilted algebras, it appears that they will have minimal length maximal green sequences of the same length (see Section 10 for more details).

The paper is organized as follows. In Section 2, we review the basics of quiver mutation, maximal green sequences, and oriented exchange graphs. We also recall the definitions of **c**- and **g**-vectors, and the result that maximal green sequences are in bijection with