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Warmth and edge spaces of graphs

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A B S T R A C T

In this paper we study a pair of numerical parameters associated to a graph G. On the one hand, we consider the (topological) connectivity of the polyhedral complex $\mathrm{Hom}(K_2, G)$, a space of homomorphisms from a edge K_2 into G. This approach dates back to the neighborhood complexes introduced by Lovász in his proof of the Kneser conjecture. In another direction we study the warmth $\zeta(G)$ of a graph G, a parameter introduced by Brightwell and Winkler based on the asymptotic behavior of d-branching walks in G and inspired by constructions in statistical physics. Both the warmth of G and the connectivity of $\mathrm{Hom}(K_2, G)$ provide lower bounds on the chromatic number of G.

Here we seek to relate these two constructions, and in particular we provide evidence for the conjecture that the warmth of a graph G is always less than three plus the connectivity of $\mathrm{Hom}(K_2, G)$. We succeed in establishing a first nontrivial case of the conjecture, by showing that $\zeta(G) \leq 3$ if $\mathrm{Hom}(K_2, G)$ has an infinite first homology group. We also calculate warmth for a family of 'twisted toroidal' graphs that are important extremal examples in the context of Hom complexes. Finally we show that $\zeta(G) \leq n - 1$ if a graph G does not have the complete bipartite graph $K_{a,b}$ for $a+b = n$.

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This provides an analogue for a similar result in the context of Hom complexes.

1. Introduction

Suppose G is a graph with no multiple edges. In recent years a pair of numerical invariants associated to G have been introduced in seemingly independent contexts. On the one hand, one can construct a *space* of homomorphisms from a edge K_2 into G and study various notions of topological *connectivity*. This construction dates back to the neighborhood complexes $N(G)$ introduced by Lovász in his proof of the Kneser conjecture, and was further developed by Babson and Kozlov. In modern treatments we recover $N(G) \simeq \text{Hom}(K_2, G)$ as a space of homomorphisms from the edge K_2 into the graph G (homomorphisms from an edge), an example of the more general Hom-complexes $\text{Hom}(T, G)$ of homomorphisms between two graphs T and G (see [1]). Precise definitions are given in Section 2, but the basic idea is that $\text{Hom}(K_2, G)$ is a polyhedral complex with 0-cells given by all directed edges of G, with higher dimensional cells given by directed complete bipartite graphs.

In another direction Brightwell and Winkler studied notions of 'long range action' of graph homomorphisms, motivated by constructions in statistical physics. They introduced a graph parameter called the *warmth* $\zeta(G)$ of a graph G, a measure of the asymptotic behavior of d-branching walks in G. The idea is a generalization of the following observation: if B is a *bipartite* graph then we can restrict the possibilities of the initial position of a random walk (thought of as a map from the one-branching tree T^1 to B) if we know where the walk is at the nth step, regardless of how large n is. The warmth $\zeta(G)$ quantifies, in a way that will be made precise in Section 2, how large d needs to be for the same to be true for maps $T^d \to G$.

It turns out the both the warmth of G and the connectivity of the edge space of G provide lower bounds on the chromatic number, by definition the fewest number of colors needed to color the vertices of G in such a way that adjacent vertices receive distinct colors. While upper bounds on chromatic number are in some sense straightforward (just 'write down a coloring') finding lower bounds often requires methods from diverse branches of mathematics. In this language, a main result of [12] is that for any graph G we have

$$\chi(G) \geq \text{conn}(\text{Hom}(K_2, G)) + 3.$$

Here we use the convention that $\text{conn}(X) = -1$ if X is nonempty and disconnected. In more recent work [2]. On the statistical physics side, Brightwell and Winkler [3] show that warmth provides a lower bound on chromatic number: For any graph G we have