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Bijective enumerations of Γ-free 0–1 matrices

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ABSTRACT

We construct a new bijection between the set of $n \times k$ 0–1 matrices with no three 1's forming a Γ configuration and the set of (n, k)-Callan sequences, a simple structure counted by poly-Bernoulli numbers. We give two applications of this result: We derive the generating function of Γ-free matrices, and we give a new bijective proof for an elegant result of Aval et al. that states that the number of complete non-ambiguous forests with n leaves is equal to the number of pairs of permutations of $\{1, \dots, n\}$ with no common rise.

1. Introduction

We call a 0–1 matrix Γ-*free* if it does not contain 1's in positions such that they form a Γ configuration; i.e. two 1's in the same row and a third 1 below the left of these in the same column. $\Gamma = \begin{smallmatrix} 1 & 1 \\ 1 & * \end{smallmatrix}$. For instance, matrix A is not Γ-free because the bold 1's form a Γ configuration, while matrix B is a Γ-free matrix.

$$A = \begin{pmatrix} 0 & \mathbf{1} & \mathbf{1} & 0 \\ 1 & 1 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

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Table 1
The poly-Bernoulli numbers $B_n^{(-k)}$.

n, k	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	4	8	16	32
2	1	4	14	46	146	454
3	1	8	46	230	1066	4718
4	1	16	146	1066	6906	41506
5	1	32	454	4718	41506	329462

Clearly, we can say that a matrix is Γ-free if and only if it does not contain any of the submatrices from the following set:

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

Pattern avoidance is an important notion in combinatorics. Matrices were also investigated from different point of view in this context; both extremal [9] and enumerative [11], [12] results are known.

Γ-free 0–1 matrices of size $n \times k$ contain at most $n + k - 1$ 1's [9]. The set of $n \times k$ 0–1 Γ-free matrices is one of the matrix classes that are enumerated by the poly-Bernoulli numbers, $B_n^{(-k)}$ [4]. Besides matrix classes that are characterized by excluded submatrices there are several other combinatorial objects that are enumerated by the poly-Bernoulli numbers. For instance, permutations with a given exceedance set, permutations with a constraint on the distance of their values and images, Callan permutations, acyclic orientations of complete bipartite graphs, non-ambiguous forests, etc. For further details, including recurrence relations and the original definition of poly-Bernoulli numbers via generating function, see [4], [5] and [6]. There is also a nice combinatorial formula of the poly-Bernoulli numbers of negative k indices: For $k > 0$,

$$B_n^{(-k)} = \sum_{m=0}^{\min(n,k)} m! \left\{ {n+1 \atop m+1} \right\} m! \left\{ {k+1 \atop m+1} \right\}, \tag{1}$$

where $\left\{ {n \atop m} \right\}$ denotes a Stirling number of the second kind. Table 1 shows the values of $B_n^{(-k)}$ for small k and n.

From (1), we give an obvious combinatorial interpretation of the numbers $B_n^{(-k)}$, which will be regarded as their combinatorial definition in this paper. (This interpretation is essentially the same as the one that counts Callan permutations.) On an (n, k)-*Callan sequence* we mean a sequence $(S_1, T_1), \ldots, (S_m, T_m)$ for some $m \in \mathbb{N}_0$ such that $S_1, \ldots, S_m$ are pairwise disjoint nonempty subsets of $\{1, \ldots, n\}$, and $T_1, \ldots, T_m$ are pairwise disjoint nonempty subsets of $\{1, \ldots, k\}$. We note that the empty sequence is also a Callan sequence with $m = 0$.

Lemma 1. *For $k > 0$, $B_n^{(-k)}$ counts the number of (n, k)-Callan sequences.*