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1324-avoiding permutations revisited

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ABSTRACT

We give an improved algorithm for counting the number of 1324-avoiding permutations, resulting in 14 further terms of the generating function, which is now known for all lengths $\leq$ 50. We re-analyse the generating function and find additional evidence for our earlier conclusion that unlike other classical length-4 pattern-avoiding permutations, the generating function does not have a simple power-law singularity, but rather, the number of 1324-avoiding permutations of length n behaves as

$$B \cdot \mu^n \cdot \mu_1^{\sqrt{n}} \cdot n^g.$$

We estimate $\mu = 11.600 \pm 0.003$, $\mu_1 = 0.0400 \pm 0.0005$, $g = -1.1 \pm 0.1$ while the estimate of B depends sensitively on the precise value of μ, μ_1 and g. This reanalysis provides substantially more compelling arguments for the presence of the stretched exponential term $\mu_1^{\sqrt{n}}$.

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1. Introduction

In an earlier paper [5], two of the current authors gave further coefficients and a detailed analysis of the generating function for 1324 pattern-avoiding permutations (PAPs), extending the known ordinary generating function (OGF) by a further 5 terms. That analysis led us to conjecture that, unlike the known length-4 PAPs, notably the classes $Av(1234)$ [13] and $Av(1342)$ [3], the OGF for $Av(1324)$ included a stretched exponential term. That is to say, if p_n denotes the number of n-step $Av(1324)$ permutations, then

$$p_n \sim B \cdot \mu^n \cdot \mu_1^{\sqrt{n}} \cdot n^g, \tag{1}$$

where estimates of the parameters were given.

In the present paper, we present a new, substantially improved algorithm that allows us to give 14 further terms.[1]

This stretched exponential behaviour is not without precedent. There are a number of models in mathematical physics whose coefficients possess this more complex asymptotic structure. In particular, Duplantier and Saleur [8] and Duplantier and David [7] studied the case of *dense* polymers in two dimensions, and found the partition functions had the asymptotic form $const \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g$. In [22], Owczarek, Prellberg and Brak investigated an exactly solvable model of interacting partially-directed self-avoiding walks (IPDSAW), and found the coefficients behaved with this asymptotic form, and estimated $\sigma = 1/2$, $g = -3/4$, while the sub-exponential growth constant μ_1 was found to more than 5 digit accuracy. From [4] the value of μ is exactly known. Subsequently Duplantier [6] pointed out that $\sigma = 1/2$ is to be expected, not only for IPDSAWs, but also for SAWs in the collapsed regime. He went on to predict the value of the exponent g in that case. For self-avoiding walks and polygons attached to a surface and pushed toward the surface by a force applied at their top vertex, Beaton et al. [1] gave probabilistic arguments for stretched exponential behaviour, but with growth $\mu_1^{n^{3/7}}$.

Such stretched exponential behaviour is also seen in other combinatorial problems. If one considers the cogrowth series of certain infinite, finitely generated amenable groups [10], one sees similar, and sometimes more complex, behaviour. For example, for the lamplighter group, the coefficients of the cogrowth series l_n behave as

$$l_n \sim const. \cdot 9^n \cdot \mu_1^{n^{1/3}} \cdot n^{1/6},$$

[24], whereas for the wreath products $\mathbb{Z} \wr \mathbb{Z}$ and $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ the coefficients behave as

$$w_n \sim const. \cdot 16^n \cdot \mu_1^{n^{1/3} \log^{2/3}(n)} \cdot n^g,$$

and

[1] The only limit to obtaining additional terms is computer memory. The present calculation required about 4.2TB of (distributed) memory.