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Neural ideals and stimulus space visualization

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ABSTRACT

A neural code $\mathcal{C}$ is a collection of binary vectors of a given length n that record the co-firing patterns of a set of neurons. Our focus is on neural codes arising from place cells, neurons that respond to geographic stimulus. In this setting, the stimulus space can be visualized as subset of $\mathbb{R}^2$ covered by a collection $\mathcal{U}$ of convex sets such that the arrangement $\mathcal{U}$ forms an Euler diagram for $\mathcal{C}$. There are some methods to determine whether such a convex realization $\mathcal{U}$ exists; however, these methods do not describe how to draw a realization. In this work, we look at the problem of algorithmically drawing Euler diagrams for neural codes using two polynomial ideals: the neural ideal, a pseudo-monomial ideal; and the neural toric ideal, a binomial ideal. In particular, we study how these objects are related to the theory of piercings in information visualization, and we show how minimal generating sets of the ideals reveal whether or not a code is 0, 1, or 2-inductively pierced.

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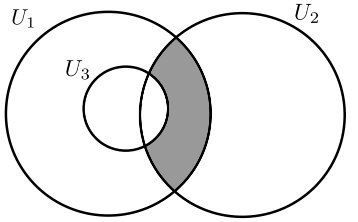

Fig. 1. An arrangement of three sets $\mathcal{U} = \{U_1, U_2, U_3\}$; the codeword associated to the shaded region is $c = 110$ and $Z_c = \text{supp}(c) = \{1, 2\}$. Here, the associated code is $\mathcal{C}(\mathcal{U}) = \{000, 100, 010, 110, 101, 111\}$.

1. Introduction

In 2014, the Nobel Prize in Medicine or Physiology was awarded to John O'Keefe and his team for their 1971 discovery of place cells [21]. A *place cell* is a neuron that codes for a distinct region in an animal's environment called a *place field*. That is, if the animal is in a place field, the associated place cell fires; otherwise it is silent. Such neurons are believed to be an essential part of the navigation system and spatial memory.

The firing activity of a population of neurons over time results in a set of co-firing patterns, which can be stored using binary vectors, or *codewords*. Each codeword indicates the set of neurons that were firing together during some time window. A set $\mathcal{C} \subset \{0, 1\}^n$ of codewords on n neurons is called a *combinatorial neural code*; the descriptor "combinatorial" is commonly used since the precise details of neural spiking and timing are discarded, leaving only discrete co-firing patterns. For a description of how neuronal firing data may be discretized, see [7].

Each codeword in a combinatorial neural code $\mathcal{C}$ is associated with the set of neurons it represents; that is, given $c \in \mathcal{C} \subset \{0, 1\}^n$, we associate c with $Z_c := \text{supp}(c) = \{i \in [n] \mid c_i = 1\}$. If the neurons in question are known to be place cells, then a codeword c of co-firing place cells indicates that the neurons in Z_c have overlapping place fields.

Place fields can be approximated by convex sets in $\mathbb{R}^2$, for example, see [6, Figure 1]. Given an arrangement of convex subsets of $\mathbb{R}^2$ representing place fields, we can easily extract the associated neural code by considering the various zones in the arrangement. That is, given a collection of sets $\mathcal{U} = \{U_1, ..., U_n\}$ with each $U_i \subset \mathbb{R}^2$ a convex set, the code associated to $\mathcal{U}$ is

$$\mathcal{C}(\mathcal{U}) = \{c \in \{0, 1\}^n \mid (\bigcap_{i \in Z_c} U_i) \setminus (\bigcup_{j \notin Z_c} U_j) \neq \emptyset\},$$

as illustrated in Fig. 1. We define $\bigcap_{i \in \emptyset} U_i = \mathbb{R}^2$ and refer to U_i as the place field of neuron i.

The inverse problem is more difficult: given a particular neural code $\mathcal{C}$ presumed to come from place cells, can we find a set of convex subsets in $\mathbb{R}^2$ which would, as place fields, exhibit $\mathcal{C}$ as its associated code? If such a collection of convex sets exists, the code is called *convexly realizable in* $\mathbb{R}^2$. Previous work [9,6,19] has considered the question of