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Alternating sign matrices and hypermatrices, and a generalization of Latin squares

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ABSTRACT

An alternating sign matrix, or ASM, is a $(0, \pm 1)$-matrix where the nonzero entries in each row and column alternate in sign. We generalize this notion to hypermatrices: an $n \times n \times n$ hypermatrix $A = [a_{ijk}]$ is an *alternating sign hypermatrix*, or ASHM, if each of its planes, obtained by fixing one of the three indices, is an ASM. Several results concerning ASHMs are shown, such as finding the maximum number of nonzeros of an $n \times n \times n$ ASHM, and properties related to Latin squares. Moreover, we investigate completion problems, in which one asks if a subhypermatrix can be completed (extended) into an ASHM. We show several theorems of this type.

1. Introduction

Let A be an $n \times n$ $(0, \pm 1)$-matrix. Then A is an *alternating sign matrix*, abbreviated ASM, provided in each of the $2n$ lines of A, that is, its rows and columns, the nonzeros alternate beginning and ending with a $+1$. Permutation matrices are ASMs without any -1's. ASMs were defined by Mills, Robbins, and Ramsey [13] and have a fascinating

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history which can be found in [2]. Extending some of the work reported in [1] and [15], we carried out in [4] a recent study of ASMs and related matrix classes and polyhedra where additional references, besides those given here, can be found. Our goal is to generalize ASMs to three-dimensional matrices called *hypermatrices*. In doing so, we were led to a fascinating generalization of classical Latin squares.

Let $A = [a_{ijk}]$ be an $n \times n \times n$ hypermatrix. We refer to i as the *row index*, j as the *column index*, and k as the *vertical index* of the hypermatrix A. Then A has three types of lines, each of cardinality n:

(i) The *row lines* (variable row index) $A_{*jk} = [a_{ijk} : i = 1, 2, \ldots n]$, $(1 \leq j, k \leq n)$;
(ii) The *column lines* (variable column index) $A_{i*k} = [a_{ijk} : j = 1, 2, \ldots n]$, $(1 \leq i, k \leq n)$;
(iii) The *vertical lines* (variable vertical index) $A_{ij*} = [a_{ijk} : k = 1, 2, \ldots n]$, $(1 \leq i, j \leq n)$.

Similarly, A has three types of planes, each of cardinality n^2:

(i) The *horizontal planes* (or *row-column planes*) (variable row and column indices) $A^{\mathrm{h}}_k = A_{**k} = [a_{ijk} : i, j = 1, 2, \ldots n]$, $(1 \leq k \leq n)$;
(ii) The *row-vertical planes* (variable row and vertical indices) $A^{\mathrm{cv}}_j = A_{*j*} = [a_{ijk} : i, k = 1, 2, \ldots n]$, $(1 \leq j \leq n)$;
(iii) The *column-vertical planes* (variable column and vertical indices) $A^{\mathrm{rv}}_i = A_{i**} = [a_{ijk} : j, k = 1, 2, \ldots n]$, $(1 \leq i \leq n)$.

The intersection of two planes of different types is a line; for instance, the intersection of a horizontal plane with a column-vertical plane is a column line:

$$A_{**k} \ \cap \ A_{i**} = A_{i*k}.$$

We usually denote the $n \times n \times n$ hypermatrix A by

$$A = [A^{\mathrm{h}}_1, A^{\mathrm{h}}_2, \ldots, A^{\mathrm{h}}_n], \text{ abbreviated to } A = [A_1, A_2, \ldots, A_n]$$

where the A_i are the horizontal planes A_{**k}. To denote the fact that A is a 3-dimensional array, we also write

$$A = A_1 \nearrow A_2 \nearrow \cdots \nearrow A_n$$

where the north-east arrow $A_i \nearrow A_{i+1}$ is read as A_i is below A_{i+1} (or A_{i+1} is on top of A_i). We can also write

$$A = [A^{\mathrm{cv}}_1, A^{\mathrm{cv}}_2, \ldots, A^{\mathrm{cv}}_n] \text{ and } A = [A^{\mathrm{rv}}_1, A^{\mathrm{rv}}_2, \ldots, A^{\mathrm{rv}}_n].$$