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Group actions on semimatroids

Emanuele Delucchi[a,*], Sonja Riedel[a,b]

[a] *Departement de Mathématiques, Université de Fribourg, chemin du musée 23, 1700 Fribourg, Switzerland*
[b] *Institute for Algebra, Geometry, Topology and Their Applications, Fachbereich Mathematik und Informatik, Universität Bremen, Bibliothekstraße 1, 28359 Bremen, Germany*

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ABSTRACT

We initiate the study of group actions on (possibly infinite) semimatroids and geometric semilattices. To every such action is naturally associated an orbit-counting function, a two-variable "Tutte" polynomial and a poset which, in the representable case, coincides with the poset of connected components of intersections of the associated toric arrangement. In this structural framework we recover and strongly generalize many enumerative results about arithmetic matroids, arithmetic Tutte polynomials and toric arrangements by finding new combinatorial interpretations beyond the representable case. In particular, we thus find a class of natural examples of nonrepresentable arithmetic matroids. Moreover, we discuss actions that give rise to matroids over $\mathbb{Z}$ with natural combinatorial interpretations. As a stepping stone toward our results we also prove an extension of the cryptomorphism between semimatroids and geometric semilattices to the infinite case.

* Corresponding author.
E-mail addresses: emanuele.delucchi@unifr.ch (E. Delucchi), sriedel@math.uni-bremen.de (S. Riedel).

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0. Introduction

This paper is about group actions on combinatorial structures. There is an extensive literature on enumerative aspects of group actions, from Pólya's classical work [32] to, e.g., recent results on polynomial invariants of actions on graphs [7]. The chapter on group actions in Stanley's book [35] offers a survey of some of the results in this vein, together with a sizable literature list. Moreover, group actions on (finite) partially ordered sets have been studied from the point of view of representation theory [33], of homotopy theory [26], and of the poset's topology [3,36].

Here we consider group actions on (possibly infinite) semimatroids and geometric semilattices from a structural perspective. We develop an abstract setting that fits different contexts arising in the literature, allowing us to unify and generalize many recent results.

Motivation. Our original motivation came from the desire to better understand the different new combinatorial structures that have been introduced in the wake of recent work of De Concini–Procesi–Vergne [14,15] on toric arrangements and partition functions, and have soon gained independent research interest. Our motivating goals are

- to organize these different structures into a unifying theoretical framework and to develop new combinatorial interpretations also in the nonrepresentable case;
- to understand the geometric side of this theory, in particular in terms of a suitable abstract class of posets (an "arithmetic" analogue of geometric lattices).

To be more precise, let us consider a list $a_1, \ldots, a_n \in \mathbb{Z}^d$ of integer vectors. Such a list gives rise to an *arithmetic matroid* (d'Adderio–Moci [9] and Brändén–Moci [5]) with an associated *arithmetic Tutte polynomial* [29], and a *matroid over the ring* $\mathbb{Z}$ (Fink–Moci [20]). Moreover, by interpreting the a_i as characters of the torus $\mathrm{Hom}(\mathbb{Z}^d, \mathbb{C}^*) \simeq (\mathbb{C}^*)^d$ we obtain a *toric arrangement* in $(S^1)^d \subseteq (\mathbb{C}^*)^d$ defined by the kernels of the characters, with an associated *poset of connected components* of intersections of these hypersurfaces. In this case, the arithmetic Tutte polynomial computes the characteristic polynomial of the arrangement's poset and the Poincaré polynomial of the arrangement's complement, as well as the Ehrhart polynomial of the zonotope spanned by the a_i and the dimension of the associated Dahmen–Micchelli space [29]. Other contexts of application of arithmetic matroids include the theory of spanning trees of simplicial complexes [17] and interpretations in graph theory [10]. After a first version of this paper was submitted, we learned about current work of Aguiar and Chan [1] focusing on toric arrangements defined by graphs. Although they stay in the "representable" realm, their interesting work refines some statistics related to arithmetic matroids and fits well into our setup.

On an abstract level, arithmetic matroids offer a theory supporting some notable properties of the arithmetic Tutte polynomial, while matroids over rings are a very general and strongly algebraic theory with different applications for suitable choices of the "base ring".