Advances in Applied Mathematics 95 (2018) 271–330

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Hopf algebras and Tutte polynomials ☆

Thomas Krajewski [a], Iain Moffatt [b,*], Adrian Tanasa [c,d,e]

[a] *CPT – UMR 7332, CNRS, Aix-Marseille Université and Université de Toulon, Campus de Luminy, 13228 Marseille Cedex 9, France*
[b] *Department of Mathematics, Royal Holloway, University of London, Egham, Surrey, TW20 0EX, United Kingdom*
[c] *Université de Bordeaux, LaBRI, UMR 5800, F-33400 Talence, France*
[d] *Horia Hulubei National Institute for Physics and Nuclear Engineering, P.O.B. MG-6, 077125 Magurele, Romania*
[e] *I. U. F., Paris, France*

A R T I C L E I N F O

Article history:
Received 20 October 2017
Accepted 7 December 2017

MSC:
05C31

Keywords:
Graph polynomial
Tutte polynomial
Hopf algebra
Las Vergnas polynomial
Penrose polynomial
Bollobás–Riordan polynomial
Krushkal polynomial

A B S T R A C T

By considering Tutte polynomials of Hopf algebras, we show how a Tutte polynomial can be canonically associated with combinatorial objects that have some notions of deletion and contraction. We show that several graph polynomials from the literature arise from this framework. These polynomials include the classical Tutte polynomial of graphs and matroids, Las Vergnas' Tutte polynomial of the morphism of matroids and his Tutte polynomial for embedded graphs, Bollobás and Riordan's ribbon graph polynomial, the Krushkal polynomial, and the Penrose polynomial.

We show that our Tutte polynomials of Hopf algebras share common properties with the classical Tutte polynomial, including deletion-contraction definitions, universality properties, convolution formulas, and duality relations. New results for graph polynomials from the literature are then obtained as examples of the general results.

Our results offer a framework for the study of the Tutte polynomial and its analogues in other settings, offering the means

☆ T.K. was partially supported by the ANR JCJC CombPhysMat2Tens grant. A.T. was partially supported by the PN 09370102 and ANR JCJC CombPhysMat2Tens grants. This work was initiated while the authors were visiting the Erwin Schrödinger International Institute for Mathematical Physics (ESI) in Vienna during their 2014 programme on Combinatorics, Geometry and Physics. All three authors would like to thank the ESI for their support and for providing a productive working environment.

* Corresponding author.
E-mail addresses: thomas.krajewski@cpt.univ-mrs.fr (T. Krajewski), iain.moffatt@rhul.ac.uk (I. Moffatt), adrian.tanasa@ens-lyon.org (A. Tanasa).

https://doi.org/10.1016/j.aam.2017.12.001

to determine the properties and connections between a wide class of polynomial invariants.

1. Introduction and overview

The Tutte polynomial is arguably the most important graph polynomial, and unquestionably the most studied. It encodes a substantial amount of the combinatorial information of a graph, specialises to a myriad of other polynomials (including the chromatic and flow polynomials). It appears in knot theory as the Jones and homfly-pt polynomials, and in statistical mechanics as the Ising and Potts model partition functions.

Given the pervasiveness of the Tutte polynomial, it is unsurprising that attention has been given to finding analogues or extensions of the Tutte polynomial from graphs to other types of combinatorial object. These analogues can mostly be fit in to three broad types. Some analogues, such as W.T. Tutte and H. Crapo's extension to matroids [15,45], uncontroversially should be called a Tutte polynomial. Some analogues, such as M. Las Vergnas' Tutte polynomial for morphisms of matroids [36,37], offer entirely satisfactory candidates for a Tutte polynomial, but without an explanation of why we should use that particular polynomial and no other. Finally, some polynomials, such as the Bollobás–Riordan polynomial [13] or G. Farr's polynomials of alternating dimaps [26], offer polynomials that have some of the properties we would expect of a Tutte polynomial, but do not have some of the other properties we would expect (for example, a "full" deletion-contraction definition in the case of the Bollobás–Riordan polynomial).

Thus we arrive at the fundamental problem of what we mean when we say that a polynomial invariant is a "Tutte polynomial" of some class of objects? It is exactly this problem that we are interested in here.

As an answer to this problem, we propose a Hopf algebraic framework for Tutte-like graph polynomials. This framework offers a canonical construction of a "Tutte polynomial" of a (suitable) set of combinatorial objects that is equipped with *some* notions of "deletion" and "contraction". (We emphasise that these need not be the usual notions of deletion and contraction for the given objects. In fact, different "Tutte polynomials" arise when using different notions of deletion and contraction for the same type of object.) The resulting polynomials satisfy what we can reasonably expect an analogue of the Tutte polynomial to:

- Have a natural, canonical definition that arises from the class.
- Have a "full" recursive deletion-contraction-type definition terminating in trivial objects.
- Have a state-sum (rank-nullity-type) formulation.