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On the complexity of generalized chromatic polynomials

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ABSTRACT

J. Makowsky and B. Zilber (2004) showed that many variations of graph colorings, called **CP**-colorings in the sequel, give rise to graph polynomials. This is true in particular for harmonious colorings, convex colorings, mcc_t-colorings, and rainbow colorings, and many more. N. Linial (1986) showed that the chromatic polynomial $\chi(G; X)$ is #**P**-hard to evaluate for all but three values $X = 0, 1, 2$, where evaluation is in **P**. This dichotomy includes evaluation at real or complex values, and has the further property that the set of points for which evaluation is in **P** is finite. We investigate how the complexity of evaluating univariate graph polynomials that arise

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from **CP**-colorings varies for different evaluation points. We show that for some **CP**-colorings (harmonious, convex) the complexity of evaluation follows a similar pattern to the chromatic polynomial. However, in other cases (proper edge colorings, mcc_t-colorings, H-free colorings) we could only obtain a dichotomy for evaluations at non-negative integer points. We also discuss some **CP**-colorings where we only have very partial results.

1. Introduction

By a classical result of R. Ladner, and its generalization by K. Ambos-Spies, [40,5], there are infinitely many degrees (via polynomial time reducibility) between **P** and **NP**, and between **P** and #**P**, provided **P** $\neq$ **NP**. In contrast to this, the complexity of evaluating partition functions or counting graph homomorphisms satisfies a dichotomy theorem: either evaluation is in **P** or it is #**P**-complete, [21,12,13]. For the definition of the complexity class #**P**, see [26] or [49].

In accordance with the literature in graph theory a finite graph $G = (V(G), E(G))$ with $n(G) = |V(G)|$ and $e(G) = |E(G)|$ has *order* $n(G)$ and *size* $e(G)$. Otherwise, the *size of a finite set* is its cardinality.

In this paper we study the *complexity of the evaluation* of generalized univariate chromatic polynomials, as introduced in [45] and further studied in [38,39]. They will be called in the sequel **CP**-colorings (for **C**ounting **P**olynomials). Among these we find:

Examples 1.1.

(i) Trivial (unrestricted) vertex colorings using at most k colors are just functions $V(G) \to [k]$. We denote by $\chi_{trivial}(G;k)$ the number of trivial colorings of G, hence $\chi_{trivial}(G;k) = k^{|V(G)|} \in \mathbb{Z}[k]$.

(ii) Proper vertex colorings using at most k colors, where two neighboring vertices receive different colors, are counted by $\chi(G;k)$, the classical chromatic polynomial.

(iii) Proper edge colorings using at most k colors, where two edges with a common vertex receive different colors, are counted by $\chi_{edge}(G;k)$, the edge chromatic polynomial. We note that they are exactly the proper vertex colorings of the line graph $L(G)$ of G.

(iv) Convex colorings using at most k colors are vertex colorings, which are not necessarily proper, but where each color class induces a connected subgraph. They are counted by $\chi_{convex}(G;k)$. Convex colorings are first introduced in [48].

(v) Harmonious colorings using at most k colors are proper vertex colorings such that no two edges have end-vertices receiving the same pair of colors. They were