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Partial graph orientations and the Tutte polynomial

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ABSTRACT

Gessel and Sagan investigated the Tutte polynomial, $T_G(x, y)$ using depth-first search, and applied their techniques to show that the number of acyclic partial orientations of a graph is $2^{m-n+1}T_G(3, 1/2)$. We provide a short deletion-contraction proof of this result and demonstrate that dually, the number of strongly connected partial orientations is $2^{n-1}T_G(1/2, 3)$. We then prove that the number of partial orientations modulo cycle reversals is $2^g T_G(3, 1)$ and the number of partial orientations modulo cut reversals is $2^{n-1}T_G(1, 3)$. To prove these results, we introduce cut and cycle-minimal partial orientations which provide distinguished representatives for partial orientations modulo cut and cycle reversals, extending known representatives for full orientations introduced by Greene and Zaslavksy. We then introduce distinguished partial orientations representing a given indegree sequence. We utilize these partial orientations to derive the Ehrhart polynomial of the win vector polytope, and give a combinatorial interpretation of its volume, thus answering a question of Bartels, Mount, and Welsh. We conclude with edge chromatic generalizations of the quantities presented, which allow for a new interpretation of the reliability polynomial for all probabilities p with $0 < p < 1/2$.

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1. Introduction

The Tutte polynomial is the most general bivariate polynomial which can be defined using deletion-contraction. There are several known evaluations of the Tutte polynomial which count certain families of graph orientations. In particular, Stanley [32] showed that $T_G(2,0)$ counts the number of acyclic orientations of a graph and Las Vergnas [25] showed that $T_G(0,2)$ counts the number of strongly connected orientations. These two facts are dual for planar graphs because acyclic orientations of a plane graph induce strongly connected orientations of its dual, while the Tutte polynomial of the dual graph is obtained by interchanging the variables. Gessel and Sagan [16] investigated the Tutte polynomial using depth-first search and applied their techniques to show the number of acyclic partial orientations of a graph is $2^{m-n+1}T_G(3,1/2)$. We offer a short proof of this fact using deletion-contraction and demonstrate for the first time that the number of strongly connected partial orientations of a graph is $2^{n-1}T_G(1/2,3)$.

Gioan [17] presented a unified framework for understanding the orientation-based interpretations of the integer evaluations of $T_G(x,y)$ for $0 \leq x, y \leq 2$ using directed cut reversals, directed cycle reversals, and a convolution formula for the Tutte polynomial in terms of the cyclic flats of a graph. In the process, Gioan introduced the notion of orientations with a quasi-sink which give distinguished representatives for orientations modulo cut reversals and are counted by $T_G(1,2)$. On the other hand, it was shown by Stanley [33] that the indegree sequences of orientations are counted by $T_G(2,1)$, and another interpretation of this quantity was given by Gioan as the number of orientations modulo cycle reversals. This quantity has yet another interpretation as the set of orientations such that the cyclic part is minimal in the sense of Greene and Zaslavsky [20, Corollary 8.2] or Bernardi [12], as these give distinguished representatives for the set of orientations modulo cycle reversals.

Using a total order on the edges and a fixed reference orientation of our graph, we introduce cut-minimal and cycle-minimal partial orientations which naturally extend Gioan's q-connected orientations and Greene and Zaslavsky's minimal orientations, respectively. Cut-minimal partial orientations are more matroidal than Gioan's q-connected orientations, even in the case of orientations, as they do not require the notion of a vertex. The cut-minimal partial orientations give unique representatives for the equivalence classes of partial orientations modulo cut reversals, which we prove are enumerated by $2^{n-1}T_G(1,3)$. Similarly, the cycle-minimal partial orientations give distinguished representatives for the equivalence classes of partial orientations modulo cycle reversals, which we prove are enumerated by $2^{m-n+1}T_G(3,1)$.

In [3], the author generalized Gioan's cycle reversal, cocycle reversal, and cycle–cocycle reversal systems to partial orientations by the addition of *edge pivots*. It was demonstrated that two partial orientations have the same indegree sequence if and only if they are equivalent by cycle reversals and edge pivots. We strengthen this result and introduce cycle-path minimal partial orientations which give distinguished representatives for the set of partial orientations with a fixed indegree sequence. We utilize cycle-path mini-

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