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Homogeneous q-partial difference equations and some applications

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ABSTRACT

In this paper, we show how to prove identities and evaluate integrals by expanding functions in terms of products of the q-hypergeometric polynomials by homogeneous q-partial difference equations, we also generalize some results of Liu (2015) [34] and Cao (2016) [14]. In addition, we generalize multilinear and multiple generating functions for the q-hypergeometric polynomials as applications. Moreover, we deduce some recurring formulas for Ramanujan's integrals, Askey–Roy integrals, Andrews–Askey integrals and moment integrals by the method of homogeneous q-partial difference equations. Finally, we build the relation of Ismail–Zhang type generating functions for the q-hypergeometric polynomials by the method of Ismail and Zhang (2016) [26].

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Ramanujan's integral
Moment integral

1. Introduction

As of today, the q-expansion formulas have been proved to be very important to the theory of basic hypergeometric series. Nevertheless, as has been pointed out by Gasper in [21] "$\cdots$ the succeeding higher order derivatives becomes more and more difficult to calculate for $|z| < 1$, and so one is forced to abandon this approach and to search for another way $\cdots$".

Polynomial solutions of q-partial differential equations and q-difference equations often serve as a building block in algorithms for finding other types of closed-form solutions, computer algebra algorithms are usual tools for finding polynomials [1,4]. The motivation for this work is to deal with the q-hypergeometric polynomials as solutions of the certain q-partial difference equations. For more information, please refer to [1,4,11,25,29–35].

In this paper, we follow the notations and terminology in [22] and suppose that $0 < q < 1$. The basic hypergeometric series ${}_r\phi_s$ defined by

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{\left(a_1, a_2, \ldots, a_r; q\right)_n}{\left(q, b_1, b_2, \ldots, b_s; q\right)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \tag{1.1} $$

converges absolutely for all z if $r \leq s$, for $|z| < 1$ if $r = s + 1$ and for terminating. The q-series and its compact factorials are defined respectively by

$$ (a; q)_0 = 1, \quad (a; q)_n = (a; q)_\infty / \left(aq^n; q\right)_\infty \tag{1.2} $$

and $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$, where $m \in \mathbb{N} := \{1, 2, 3, \cdots\}$ and $n \in \mathbb{C}$. The binomial coefficient is defined by

$$ \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}. $$

The Rogers–Szegö polynomials were introduced by Szegö in 1926 but were already studied earlier by Rogers in 1894–1895. A good definition can be found in the book by Simon [39, Ex. (1.6.5), pp. 77–87].

The homogeneous Rogers–Szegö polynomials are defined by [34, p. 3]

$$ h_n(b, c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} b^k c^{n-k} \quad \text{and} \quad g_n(b, c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} b^k c^{n-k}. \tag{1.3} $$

The Al-Salam–Carlitz polynomials are defined by Al-Salam and Carlitz in 1965 [2, Eqs. (1.11) and (1.15)]