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Coincidence for morphisms based on compactness conditions on countable sets

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ABSTRACT

We present general coincidence results for morphisms satisfying certain compactness type conditions on countable sets. Our theory is based on coincidence principles for compact morphisms.

1. Introduction

Morphisms (Vietoris fractions) in the sense of Gorniewicz and Granas was introduced in [4] and discussed in detail in the books [3,6] and in the papers [1,4,5,8,9]. In this paper motivated by Mönch's fixed point principle [7,10,11] we present two general coincidence results for morphisms (satisfying some compactness type condition on countable sets) defined on metrizable topological vector spaces. Our theory is based on coincidence principles for compact morphisms. This paper is a companion to our paper [9] where another approach is presented. Also we present a coincidence result for strongly admissible maps in the sense of Gorniewicz [3] at the end of this paper.

Now we present some ideas needed in Section 2. Let H be the Cěch homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Cěch homology group with compact carriers of X. For a continuous map $f\colon X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q}\colon H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p\colon \Gamma \to X$ is called a Vietoris map (written $p\colon \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii) p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $D(X, Y)$ be the set of all pairs $X \stackrel{p}{\Leftarrow} \Gamma \stackrel{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p, q). Given two diagrams (p, q) and (p', q'), where $X \stackrel{p'}{\Leftarrow} \Gamma' \stackrel{q'}{\to} Y$, we write $(p, q) \sim (p', q')$ if there are continuous maps $f\colon \Gamma \to \Gamma'$ and $g\colon \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p, q) \in D(X, Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \stackrel{p}{\Leftarrow} \Gamma \stackrel{q}{\to} Y\} : X \to Y$$

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or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X, Y)$ be the set of all such morphisms. Note if $(p, q), (p_1, q_1) \in D(X, Y)$ (where $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} Y$ and $X \overset{p_1}{\Leftarrow} \Gamma' \overset{q_1}{\rightarrow} Y$) and $(p, q) \sim (p_1, q_1)$ then it is easy to see (use $q \circ g = q_1$ and $p \circ g = p_1$ where $g\colon \Gamma' \to \Gamma$) that for $x \in X$ we have $q_1\,(p_1^{-1}(x)) = q\,(p^{-1}(x))$. For any $\phi \in M(X, Y)$ a set $\phi(x) = q\,p^{-1}\,(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ. Let $\phi \in M(X, Y)$ and (p, q) a representative of ϕ. We define $\phi(X) \subseteq Y$ by $\phi(X) = q\,(p^{-1}(X))$. Note $\phi(X)$ does not depend on the representative of ϕ. Now $\phi \in M(X, Y)$ is called compact provided the set $\phi(X)$ is relatively compact in Y. Note we will identify a map $f\colon X \to Y$ with the morphism $f = \{X \overset{Id_X}{\Leftarrow} X \overset{f}{\rightarrow} Y\} : X \to Y$. Let $X \subseteq Y$. A point $x \in X$ is called a fixed point of a morphism $\phi \in M(X, Y)$ if $x \in \phi(x)$.

Let $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} Y\} : X \to Y$ be a morphism. We define the coincidence set

$$Coin\,(p,q) = \{y \in \Gamma :\ p(y) = q(y)\}.$$

We say ϕ has a coincidence provided the set $C(\phi) = p\,(Coin\,(p,q))$ is nonempty (i.e. there exists $x \in p\,(Coin\,(p,\ q))$ i.e. there exists $y \in \Gamma$ with $x = p(y) = q(y)$). Let $(p',\ q')$ be another representation of ϕ, say $\phi = \{X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\rightarrow} Y\}$. Note $p\,(Coin\,(p,q)) = p'\,(Coin\,(p',q'))$; to see this note if $x \in p\,(Coin\,(p,\ q))$ then $x = p(y) = q(y)$ for some $y \in \Gamma$ and now since $(p,\ q) \sim (p',\ q')$ and with $f\colon \Gamma \to \Gamma'$ we have $x = q(y) = q'(f(y))$ and $x = p(y) = p'(f(y))$ so $f(y) \in \Gamma'$ and $x = q'(f(y)) = p'(f(y))$ i.e. $x \in p'\,(Coin\,(p',\ q'))$. Thus the above definition does not depend on the choice of a representation (p, q). Also $C(\phi) \neq \emptyset$ iff $Coin\,(p,\ q) \neq \emptyset$ for any representation (p, q) of ϕ.

Suppose $\phi \in M(X, X)$ (here $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} X\}$) has a coincidence point for (p, q) i.e. suppose there exists $y \in \Gamma$ with $p(y) = q(y)$. Now since p is surjective there exists $w \in X$ with $y \in p^{-1}(w)$ (note $p(w) = y$) and so $w \in q\,(p^{-1}(w)) = \phi(w)$ (note $p\,p^{-1}(w) = w$ and the set $q\,(p^{-1}(w))$ is the image of w under ϕ) i.e. ϕ has a fixed point. As a result

$$p(y) = q(y),\ y \in \Gamma\ (\text{and let}\ w = p(y)) \Leftrightarrow w \in q\,(p^{-1}(w));$$

note if $w \in q\,(p^{-1}(w))$ then there exists $y \in p^{-1}(w)$ with $w = q(y)$ so $p(y) \in p\,p^{-1}(w) = w$ (i.e. $p(y) = w$) and so $p(y) = q(y)$. In particular if the morphism $\phi \in M(X, X)$ (here (p, q) is a representation of ϕ) has a fixed point (say w i.e. $w \in q\,(p^{-1}(w))$) then there exists $y \in p^{-1}(w)$ with $q(y) = p(y)$, so ϕ has a coincidence point for (p, q), and now since we can do this argument for any representation (p, q) of ϕ (recall if (p_1, q_1) is another representation of ϕ then $(p, q) \sim (p_1, q_1)$ and as above $q\,(p^{-1}(w)) = q_1\,(p_1^{-1}(w))$ so $w \in q_1\,(p_1^{-1}(w))$ so there exists $y_1 \in p_1^{-1}(w)$ with $q_1(y_1) = p_1(y_1)$) then $Coin\,(p,\ q) \neq \emptyset$ for any representation (p, q) of ϕ i.e. ϕ has a coincidence.

2. Coincidence theory

We begin immediately with our first main result.

Theorem 2.1. *Let X be a metrizable topological vector space, $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} X\} \in M(X,X)$ (here Γ is a Hausdorff topological space) and $x_0 \in p(\Gamma)$. Assume the following conditions hold:*

$$\begin{cases} A \subseteq \Gamma,\ \ A = p^{-1}(\overline{co}\,(\{x_0\} \cup q(A)))\ \text{with}\ C \subseteq A \\ \text{countable and}\ p(C) \subseteq \overline{co}\,(\{x_0\} \cup q(C)), \\ \text{implies}\ \overline{co}\,(q(C))\ \text{is compact} \end{cases} \tag{2.1}$$

and

$$\begin{cases} \text{for any nonempty convex compact subset}\ K\ \text{of}\ X\ \text{and} \\ \text{any}\ \psi \in M(K,K)\ \text{we have that}\ \psi\ \text{has a coincidence.} \end{cases} \tag{2.2}$$

Then ϕ has a coincidence.

Remark 2.2. In the proof below we see that X metrizable can be replaced by any space with the following properties: (i). X is such that the closure of a subset Ω of X is compact if and only if Ω is sequentially compact, and (ii). for any convex set $D \subseteq X$ if $x \in \overline{D}$ then there exists a sequence $x_1, x_2, \ldots$ in D with x_n converging to x.

Remark 2.3. In (2.1) in fact $\overline{co}\,(q(C))$ is compact implies $\overline{co}\,(q(A))$ is compact (see the proof below).

Remark 2.4. Conditions to guarantee (2.2) can be found in [4,8].

Proof. Let $\mathcal{F}$ be the family of all subsets D of Γ with $p^{-1}(\overline{co}\,(\{x_0\} \cup q(D))) \subseteq D$. Note $\mathcal{F} \neq \emptyset$ since $\Gamma \in \mathcal{F}$ (recall p is surjective). Let

$$D_0 = \cap_{D \in \mathcal{F}} D\ \text{and}\ D_1 = p^{-1}(\overline{co}\,(\{x_0\} \cup q(D_0))).$$

We now show $D_1 = D_0$. Now for <u>any</u> $D \in \mathcal{F}$ we have since $D_0 \subseteq D$ that

$$D_1 = p^{-1}(\overline{co}\,(\{x_0\} \cup q(D_0))) \subseteq p^{-1}(\overline{co}\,(\{x_0\} \cup q(D))) \subseteq D,$$

so as a result $D_1 \subseteq D_0$. Also since $D_1 \subseteq D_0$ we have $q(D_1) \subseteq q(D_0)$ so

$$p^{-1}(\overline{co}\,(\{x_0\} \cup q(D_1))) \subseteq p^{-1}(\overline{co}\,(\{x_0\} \cup q(D_0))) = D_1,$$