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# Reproducing kernel method for the numerical solution of the 1D Swift–Hohenberg equation

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ABSTRACT

The Swift–Hohenberg equation is a nonlinear partial differential equation of fourth order that models the formation and evolution of patterns in a wide range of physical systems. We study the 1D Swift–Hohenberg equation in order to demonstrate the utility of the reproducing kernel method. The solution is represented in the form of a series in the reproducing kernel space, and truncating this series representation we obtain the *n*-term approximate solution. In the first approach, we aim to explain how to construct a reproducing kernel method without using Gram-Schmidt orthogonalization, as orthogonalization is computationally expensive. This approach will therefore be most practical for obtaining numerical solutions. Gram-Schmidt orthogonalization is later applied in the second approach, despite the increased computational time, as this approach will prove theoretically useful when we perform a formal convergence analysis of the reproducing kernel method for the Swift–Hohenberg equation. We demonstrate the applicability of the method through various test problems for a variety of initial data and parameter values.



## 1. Introduction

The Swift–Hohenberg equation is a nonlinear partial differential equation of fourth order which has been widely used as a model for the study of pattern formation. It was first put forward by Swift and Hohenberg [25] as a simple model for the Rayleigh-Bernard instability of roll waves. Since then, the Swift–Hohenberg equation has proved an effective model equation for a variety of phenomena in physics and mechanics. Details of the physics of the Swift–Hohenberg equation can be found in [8,9,24,26]. The Swift–Hohenberg equation is defined as [2]

$$\frac{\partial u}{\partial t} = \lambda u - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u - u^3, \tag{1}$$

where $\lambda \in \mathbb{R}$ is a parameter. The Swift–Hohenberg equation is a model equation for a large class of higher-order parabolic model equations. It has a great deal of application, such as the extended Fisher–Kolmogorov equation in statistical mechanics [2,14,32] and a sixth-order equation introduced by Caginalp and Fife [7] in phase field models [15]. Writing Eq. (1) in a

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more conventional form, we have

$$\frac{\partial u}{\partial t}+\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^2 u}{\partial x^2}+(1-\lambda)u+u^3=0. \tag{2}$$

We choose the problem domain $x\in[0,\ 1]$ and $t>0$, along with the boundary and initial conditions

$$u(0,t)=u(1,t)=0,\quad u_{xx}(0,t)=u_{xx}(1,t)=0,\quad u(x,0)=f(x). \tag{3}$$

Reproducing kernels appeared in the work of Zaremba on boundary value problems involving harmonic and biharmonic functions [31]. Bergman attempted to provide a fundamental framework for the theory of reproducing kernels [5,6], while Aronszajn produced a systematic reproducing kernel space method based on Bergman's works [3,4]. Reproducing kernel methods have proven useful in many areas, including statistics and machine learning, and they play a valuable role in complex analysis, nonlinear system of boundary value problems, nonlinear initial value problems, singular nonlinear two-point periodic boundary value problems and singularly perturbed turning point problems, probability, group representation theory, and the theory of integral operators [1,11–13,16,17,19–21,28,29].

The aim of this paper is to introduce a numerical technique based on reproducing kernel Hilbert space methods in order to solve the Swift–Hohenberg initial-boundary value problem (2)–(3). This remainder of this paper is organized as follows. In Section 2, we give a brief introduction to reproducing kernel Hilbert spaces. In Section 3, we present two specific reproducing kernel methods to solve the Swift–Hohenberg initial-boundary value problem. In the first method, we employ non-orthogonal basis functions, while in the second method we using orthogonal basis functions by way of Gram-Schmidt orthogonalization. In Section 4, we provide a convergence analysis of reproducing kernel methods for the Swift–Hohenberg initial-boundary value problem, using the second method. Numerical examples are provided in Section 5 to demonstrate the effectiveness of the proposed method. Concluding remarks, and suggestions for future work, are given in Section 6.

## 2. Reproducing kernel method preliminaries

To solve the boundary value problem (2)–(3) using the reproducing kernel theory, we need to first discuss some preliminary results.

In order to use the reproducing kernel space to solve (2)–(3), we need to first homogenize the problem. To begin, we define $u(x,t)=\vartheta(x,t)+f(x)$, and consequently we can rewrite (2)–(3) as

$$\begin{aligned}&\frac{\partial\vartheta}{\partial t}+\frac{\partial^4\vartheta}{\partial x^4}+f^{(4)}(x)+2\frac{\partial^2\vartheta}{\partial x^2}+2f''(x)+(1-\lambda)(\vartheta(x,t)+f(x))+[\vartheta(x,t)+f(x)]^3=0,\\&\vartheta(0,t)=\vartheta(1,t)=0,\quad \vartheta_{xx}(0,t)=\vartheta_{xx}(1,t)=0,\quad \vartheta(x,0)=0.\end{aligned} \tag{4}$$

We shall consider the problem domain $\Phi=[0,1]\times[0,\infty)$.

We now present some necessary definitions and theorems in the theory of reproducing kernel spaces. A Hilbert space $\omega$ of functions is called a reproducing kernel Hilbert space if there exists a reproducing kernel $R$ of $\omega$. The existence of the reproducing kernel of a Hilbert space is due to the Riesz Representation Theorem, which states that any continuous linear functional can be represented by an inner product with a unique element of space. Therefore, it is known that the reproducing kernel is unique. Note that the reproducing kernel, $R$, has the following reproducing property

$$u(\cdot)=\langle u(x),R(x,\cdot)\rangle_{\omega}\qquad \text{for all}\quad u\in\omega. \tag{5}$$

Consider the following reproducing kernel space ${}^0\omega_2^2([0,\infty))$, which is defined as

$$\begin{aligned}{}^0\omega_2^2([0,\infty))=\{\upsilon(t)\,|\,\upsilon(t),\upsilon'(t)\ &\text{are absolutely continuous functions},\\ &\upsilon,\upsilon'(t),\upsilon''(t))\in L^2([0,\infty)),\upsilon(0)=0\}.\end{aligned}$$

The name *reproducing kernel* is motivated by the reproducing property, which is evident in the action of taking the inner product. The inner product and norm for ${}^0\omega_2^2([0,\infty))$ are given by

$$\langle\upsilon_1,\upsilon_2\rangle_{{}^0\omega_2^2[0,\infty)}=\int_0^\infty\left\{4\upsilon_1(t)\upsilon_2(t)+5\upsilon_1'(t)\upsilon_2'(t)+\upsilon_1''(t)\upsilon_2''(t)\right\}\mathrm{d}t$$

and

$$\|\upsilon\|_{\omega_2^2}=\sqrt{\langle\upsilon,\upsilon\rangle_{\omega_2^2}},$$

respectively. By using the definition of the inner product and the reproducing property (5), we can write the following