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Polychromatic colorings and cover decompositions of hypergraphs

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ABSTRACT

A polychromatic coloring of a hypergraph is a coloring of its vertices in such a way that every hyperedge contains at least one vertex of each color. A polychromatic m-coloring of a hypergraph H corresponds to a cover m-decomposition of its dual hypergraph H^*. The maximum integer m that a hypergraph H admits a cover m-decomposition is exactly the longest lifetime for a wireless sensor network (WSN) corresponding to the hypergraph H. In this paper, we show that every hypergraph H has a polychromatic m-coloring if $m \leq \lfloor \frac{S}{\ln(c\Delta S^2)} \rfloor$, where $0 < c < 1$, and $\Delta \geq 1$, $S \geq 2$ are the maximum degree, the minimum size for all hyperedges in H, respectively. This result improves a result of Henning and Yeo on polychromatic colorings of hypergraphs in 2013, and its dual form improves one of Bollobás, Pritchard, Rothvoß, and Scott on cover decompositions of hypergraphs in 2013. Furthermore, we give a sufficient condition for a hypergraph H to have an "equitable" polychromatic coloring, which extends the result of Henning and Yeo in 2013 and improves in part one of Beck and Fiala in 1981 on 2-colorings (property B) of hypergraphs.

1. Introduction

In this paper, we study two interrelated problems, polychromatic colorings and cover compositions of hypergraphs. The former is a generalization of 2-colorings (Property B) of hypergraphs, and the latter is closely related to the problem of maximizing the lifetime of coverage of targets in a wireless sensor network (WSN) with battery-limited sensors.

A *hypergraph* $H = (V, \mathcal{E})$ consists of a ground set V of *vertices* and a collection $\mathcal{E}$ of *hyperedges*, where each hyperedge $E \in \mathcal{E}$ is a subset of V. Throughout this paper, we consider finite hypergraphs $H = (V, \mathcal{E})$. In order to define "dual" and "shrinking" to be referred in Sections 2 and 3 (see Remark 1 and the proofs of Theorems 1.7 and 1.9), we permit $\mathcal{E}$ to contain multiple copies of the same subset of V and also allow hyperedges of size 0, 1 and vertices of degree 0, 1. Later, the reader will be shown that the cases for the hypergraphs containing hyperedges of size 0 or 1, or vertices of degree 0 or 1 are trivial for the two problems to be discussed. The *rank* of a hypergraph H is $R(H) = \max_{E\in\mathcal{E}} |E|$, the *anti-rank* of H is $S(H) = \min_{E\in\mathcal{E}} |E|$. If $R(H) = S(H) = k$, that is, the size of every hyperedge in H would always be k, we say that the hypergraph H is a *k-uniform hypergraph*. The *degree* of a vertex $v \in V(H)$ is the number of hyperedges containing v in H, and is denoted by $d_H(v)$ or simply by $d(v)$. A hyperedge E in H is called *isolated* if it does not intersect any hyperedge of H; in particular, when $E \neq \emptyset$, each $v \in E$ has degree 1. A vertex $v \in V$ is *isolated*, if v is not contained by any hyperedge of the hypergraph. Clearly, an isolated vertex in a hypergraph has degree 0. The *maximum degree* of H is denoted by $\Delta(H) = \max_{v\in V(H)} d_H(v)$,

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the *minmum degree* of H is denoted by $\delta(H)=\min_{v\in V(H)} d_H(v)$. A hypergraph in which all vertices have the same degree k is called a *k-regular hypergraph*. Throughout this paper, we denote the class of *k-regular k-uniform hypergraphs* by $\mathcal{H}_k$.

A hypergraph H is 2-*colorable* (or has *Property B*) if it has a 2-coloring of the vertices with no monochromatic hyperedges. Alon and Bregman [1] established the following result.

Theorem 1.1 *[1]. Every hypergraph in $\mathcal{H}_k$ is 2-colorable, provided that $k\ge 8$.*

Furthermore, Vishwanathan [8] obtained the following result. Henning and Yeo [7] provided a short proof.

Theorem 1.2 *[8]. Every hypergraph in $\mathcal{H}_k$ is 2-colorable, provided that $k\ge 4$.*

Noting that Fano plane is in $\mathcal{H}_3$ and but not 2-colorable, the bound above for k is sharp.

Chen et al. [5] discussed a class of 2-colorings of hypergraphs in which each color appears at least two times on each hyperedge.

Theorem 1.3 *[5]. Let H be a hypergraph in which every hyperedge contains at least k ($k\ge 4$) vertices and meets at most d other hyperedges. If $e(k+1)2^{-k}(d+2)\le 1$, then H has a 2-coloring such that each hyperedge contains at least two vertices of each color.*

The below result, due to Henning and Yeo [7], shows us some bounds for k when we want each color to appear at least $r+1$ times on each hyperedge of a hypergraph in $\mathcal{H}_k$.

Theorem 1.4 *[7]. Let $k\ge 2$, $r\ge 2$ be two integers and $H\in\mathcal{H}_k$. Then H has a 2-coloring such that each hyperedge contains at least $r+1$ vertices of each color if one of the following conditions holds.*

(a) $r\le k/2-\sqrt{k\ln(k\sqrt{2e})}$.
(b) $k\ge 2r+3\sqrt{r\ln(r)}+44.03$.
(c) $k\ge 2r+4\sqrt{r\ln(r)}+14.04$.

Here conditions (a)–(c) imply that $k\ge 24$, $k\ge 52$, $k\ge 23$, respectively.

A result of Beck and Fiala [2] is related to the problem to partition the vertices of a hypergraph into two parts in such a way that each part is relatively "equitable" on each hyperedge.

Theorem 1.5 *[2]. Let H be a hypergraph with maximum degree Δ, where $\Delta\ge 2$. Then, H has a 2-coloring such that each hyperedge $E\in\mathcal{E}$ contains at least $|E|/2-\Delta+1$ vertices of each color.*

Let m be a positive integer. An m-coloring of the vertices of a hypergraph is a *polychromaticm-coloring* if every hyperedge contains at least one vertex of each color. This is clearly a generalization of 2-colorings of a hypergraph. A natural question is how large m could be if we want a hypergraph to have a polychromatic m-coloring. Henning and Yeo [7] gave the following result.

Theorem 1.6 *[7]. Let $k\ge 2$, $m\ge 2$ be two integers. If $m\le \frac{k}{\ln(k^3)}$, then every hypergraph $H\in\mathcal{H}_k$ has a polychromatic m-coloring.*

Obviously, if a hypergraph H has a polychromatic m-coloring, then H has a polychromatic l-coloring for each $1\le l\le m$. Therefore, it is interesting to find the maximum m that the hypergraph H admits a polychromatic m-coloring, which is called the *polychromatic number* of H and denoted by $p(H)$. So Theorem 1.6 means that $p(H)\ge \lfloor\frac{k}{\ln(k^3)}\rfloor$ for each $H\in\mathcal{H}_k$ ($k\ge 2$). Trivially, $p(H)=1$ for each hypergraph H with anti-rank 1. Next, we concentrate on the hypergraphs with anti-rank at least 2. One of our main results is below.

Theorem 1.7. *Let $S\ge 2$ and $\Delta\ge 1$ be two integers, and let H be a hypergraph with maximum degree at most Δ and anti-rank at least S. Then $p(H)\ge\lfloor\frac{S}{\ln(c\Delta S^2)}\rfloor$, where $c=\frac{e+1}{\ln(e\Delta S^2)}$.*

Note that $0<c<1.5582<e$ here. Immediately, we have the following result for hypergraphs H in $\mathcal{H}_k$.

Corollary 1.8. *Let H be a hypergraph in $\mathcal{H}_k$ and $k\ge 2$. Then $p(H)\ge\lfloor\frac{k}{\ln(ck^3)}\rfloor$, where $c=\frac{e+1}{1+\ln(k^3)}$.*

When $k\ge 15$, we have $\frac{k}{\ln(ck^3)}\ge 2$ and $0<c<0.41$. Note that the conditions in Theorem 1.6 implies that $k\ge 17$. Thus, Corollary 1.8 improves Theorem 1.6.

Furthermore, we obtain a result on "equitable" polychromatic colorings of hypergraphs.

Theorem 1.9. *Let $S\ge 2$ and $\Delta\ge 1$ be two integers, and let H be a hypergraph with maximum degree at most Δ and anti-rank at least S. Then, for every positive integers $m\le\frac{S}{\ln(e\Delta S^2)}$ and $r=\lfloor\ln(e\Delta S^2)\rfloor$, H has a polychromatic m-coloring such that every hyperedge in H contains at least r vertices of each color.*

For each fixed Δ, $\frac{S}{\ln(e\Delta S^2)}\ge 2$ when S is large enough. So this result improves Theorem 1.5 under the following conditions: (a) $m=2=\lfloor\frac{S}{\ln(e\Delta S^2)}\rfloor$; (b) the gap between the rank and the anti-rank of a hypergraph is less than $2\Delta-4$. In