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Idea of invariant subspace combined with elementary integral method for investigating exact solutions of time-fractional NPDEs

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ABSTRACT

In this paper, inspired by the idea of invariant subspace method and combined with elementary integral method, we introduced a novel approach for investigating exact solutions of a time-fractional nonlinear partial differential equation (NPDE). Based on hypothetical structure of solution of separated variable, a time-fractional NPDE defined by time and space variables can be reduced to a nonlinear ordinary differential equation (NODE) or NODEs defined by space variable alone, and then using the elementary integral method to solve the NODE or NODEs, different kinds of exact solutions of a time-fractional NPDE are obtained finally. As examples, the time-fractional Hunter–Saxton equation and time-fractional Li–Olver equation were studied. Different kinds of exact solutions of these equations were obtained and their dynamical properties were illustrated.

1. Introduction

Since the conception of fractional derivative was discussed by Hospital and Leibniz in their letters in 1695, a number of definitions of the fractional derivative such as Riemann–Liouville definition, Grünwald–Letnikov definition, Caputo definition, Riez–Feller definition, Miller–Ross sequential definition, Weyl definition and Jumarie's definition have emerged. Fractional differential equations which are established by above definitions of the fractional derivative have been applied in various fields such as viscoelastic flow [1–4], signal processing [5–7], control systems [8,9], material diffusion including normal diffusion and anomalous diffusion (super-diffusion and sub-diffusion, slow diffusion and fast diffusion) [10–13], biology systems [14,15], magnetohydrodynamics (MHD) [16–18], and many other fields. In recent decades, fractional nonlinear partial differential equations (NPDEs) have been more and more widely followed with interest due to they can be used to accurately describe nonlinear phenomena. Especially, many natural phenomena in connection with memory not only depend on time instant but also depend on the previous time history, they can be successfully modeled by the time-fractional NPDEs. Therefore, searching exact solutions and approximate analytical solutions of fractional NPDEs have extensive applications in many research fields.

In recent years, many effective methods were used to solve fractional differential equations, these methods include Adomian decomposition method [19,20], first integral method [21], homotopy analysis method [22,23], Lie group theory method [24,25], invariant subspace method [26–28], fractional variational method [29–31], method of fractional complex transformation [32–35], and so forth. Although exact solutions or approximate analytical solutions of some fractional NPDEs can

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be successfully obtained by using above methods, this is far from enough, new theories and methods for searching exact solutions of fractional NPDEs are still required to develop unceasingly because some methods have limitations on solving more complex fractional NPDEs.

Among above mentioned methods, we especially notice that some authors [32–35] employed a fractional complex transformation

$$u(x,t)=U(\xi),\quad \xi=\frac{px^{\beta}}{\Gamma(1+\beta)}+\frac{qx^{\alpha}}{\Gamma(1+\alpha)} \tag{1.1}$$

and the Jumarie's fractional chain rule [36–38]

$$D_x^{\alpha}f[u(x)]=f'(u)D_x^{\alpha}u(x)=D_u^{\alpha}f(u)(u'(x))^{\alpha} \tag{1.2}$$

to reduce the following fractional partial differential equation

$$P(u,D_t^{\alpha}u,D_x^{\beta}u,D_t^{2\alpha}u,D_t^{\alpha}D_x^{\beta}u,D_x^{2\beta},\dots,)=0 \tag{1.3}$$

into an ordinarily differential equation of integer order

$$Q(U,U',U'',U''',\dots)=0, \tag{1.4}$$

thereby they obtained many exact solutions of compound function type of a series of fractional NPDEs formed as Eq. (1.3). In addition, the Jumarie's calculus can be applied to non-differential functions, it is quite consistent with quantum mechanics and can be explained the chain rule properties, see the introductions in [39,40]. However, it should be noted that although the fractional chain rule (1.2) holds under the Jumarie's calculus definition, it does not hold under the original Riemann–Liouville definition and Caputo definition, which has been successively verified in Refs. [41–43]. So, we cannot use the fractional chain rule (1.2) to search exact solutions of those fractional NPDEs defined by Riemann–Liouville derivative and Caputo derivative. In other words, we should pay attention to the scope of application of formula (1.2) in the process of solving fractional NPDEs.

Different from the method of fractional complex transformation, recently some authors [24–28] gave a very effective method which is called invariant subspace method. This new method can be successfully used to search exact solutions of time-fractional NPDEs. For examples, in Ref. [26], by using this method, Sahadevan and Bakkyaraj investigated exact solutions of the following nonlinear time-fractional KdV equation

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\hat{F}[u]=\frac{\mu}{2}\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial^3 u}{\partial x^3}\right),\quad \alpha\in(0,1)-1/2. \tag{1.5}$$

Sahadevan and Bakkyaraj fund that the Eq. (1.5) has an invariant subspace $W_3=\mathfrak{L}\{1,x,x^2\}$ by verifying $\hat{F}[C_1+C_2x+C_3x^2]\in W_3$. Employing this invariant subspace $W_3=\mathfrak{L}\{1,x,x^2\}$, they supposed that Eq. (1.5) has the following type of solution

$$u(x,t)=a(t)+b(t)x+c(t)x^2, \tag{1.6}$$

where $a=a(t),\ b=b(t),\ c=c(t)$ are unknown functions to be determined. Substituting (1.6) into (1.5) and letting the power exponents of $x^j,\ (j=0,1,2)$ to be zero, they obtained the following time-fractional ODEs

$$\frac{d^{\alpha}a}{dt^{\alpha}}=\frac{\mu b^2}{2},\quad \frac{d^{\alpha}b}{dt^{\alpha}}=2\mu bc,\quad \frac{d^{\alpha}c}{dt^{\alpha}}=2\mu c^2.$$

Solving above time-fractional ODEs, they obtained

$$c(t)=\frac{1}{2\mu}\frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)}t^{-\alpha},\quad b(t)=t^{-\alpha},\quad a(t)=\frac{\mu}{2}\frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)}t^{-\alpha},\quad \alpha\neq\frac{1}{2}.$$

Thus they obtained an exact solution of (1.5) as follows:

$$u(x,t)=\left[\frac{\mu}{2}\frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)}+x+\frac{1}{2\mu}\frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)}x^2\right]t^{-\alpha},\quad \alpha\in(0,1)-1/2. \tag{1.7}$$

Through in-depth study for the invariant subspace method, we naturally ask: whether are there other types of invariant subspaces defined by non-elementary functions and how can we get them? How can we directly obtain more invariant subspaces except the way such as verifying and testing $\hat{F}[C_1+C_2x+C_3x^2]\in\mathfrak{L}\{1,x,x^2\}$? In other words, how can we directly obtain more functions which can be composed of invariant subspace without help of the invariant subspace method? Indeed, in Ref. [43], bypassing the invariant subspace method, we offered a new way named homogenous balanced principle to investigate exact solutions of a series of time-fractional NPDEs. In this paper, inspired by the idea of invariant subspace method but is different from this method, we will introduce another new way for searching exact solutions of time-fractional NPDE formed as the follow:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\hat{F}\left(u,\ \frac{\partial u}{\partial x},\ \frac{\partial^2 u}{\partial x^2},\ \dots,\ \frac{\partial^n u}{\partial x^n}\right),\quad 0<\alpha<1, \tag{1.8}$$