Applied Mathematics and Computation 339 (2018) 172–185

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the fault-tolerant metric dimension of convex polytopes

Hassan Raza[a], Sakander Hayat[b,*], Xiang-Feng Pan[a]

[a] School of Mathematical Sciences, Anhui University, Hefei 230601, PR China
[b] School of Mathematical Sciences, University of Science and Technology of China (USTC), Hefei 230026, PR China

ARTICLE INFO

JEL classification:
05C12
05C90

Keywords:
Metric dimension
Fault-tolerant metric dimension
Convex polytopes

ABSTRACT

A convex polytopes is a polytope that is also a convex set of points in the n-dimensional Euclidean space $\mathbb{R}^n$. By preserving the same adjacency relation between vertices of a convex polytope, its graph is constructed. The metric dimension problem has been extensively studied for convex polytopes and other families of graphs. In this paper, we study the fault-tolerant metric dimension problem for convex polytopes. By using a relation between resolving sets and fault-tolerant resolving sets of graphs, we prove that certain infinite families of convex polytopes are the families of graphs with constant fault-tolerant metric dimension. We conclude the paper with some open problems.

1. Introduction

For undefined notations and terminologies, we refer the readers to Section 2.

The concept of metric dimension was introduced independently by Slater [29] and Harary and Melter [14] in 1975 and 1976, respectively. This problem has been investigated widely since then. The metric dimension has a lot of applications in different areas of science and technology. The concept of the fault-tolerant metric dimension is a recent development in this line of research. Later in this section, we will discuss some of its applications in other areas.

The metric dimension arises in many diverse areas, including telecommunications [5], connected joints in graphs and chemistry [10], the robot navigation [24] and geographical routing protocols [26], etc. In the area of telecommunication, it is interesting to see the metric dimension problem application to network discovery and verification [5].

Metric dimension of several interesting classes of graphs have been investigated: Johnson and Kneser graph [3], Grassmann graphs [4], Cayley digraphs [11] and cartesian product of graphs [8]. Siddiqui et al. [28] investigated the metric dimension of some infinite families of wheel-related graphs. Kratica et al. [25] studied the metric dimension problem of certain convex polytopes. Imran et al. [21] studied further the metric dimension of convex polytopes generated by wheel-related graphs. In [7], it has been shown that the metric dimension of a graphs is not necessarily a finite natural number. They proved that some infinite graphs have infinite metric dimension. The computational complexity of this problem is studied in [13].

Elements of metric basis were referred to as censors in an application given in [9]. If one of the censors does not work properly, we will not have enough information to deal with the intruder (fire, thief, etc). In order to overcome these kind of problems, concept of the fault-tolerant metric dimension was introduced by Hernando et al. [17]. A fault-tolerant resolving set provides correct information even when one of the censors is not working. Consequently, fault-tolerant resolving sets

* Corresponding author.
E-mail addresses: hassan_raza783@yahoo.com (H. Raza), sakander@mail.ustc.edu.cn (S. Hayat), xfpan@ahu.edu.cn (X.-F. Pan).

https://doi.org/10.1016/j.amc.2018.07.010
0096-3003/

are important from the applicability perspective. We refer the readers to [27] for more applications of metric dimension and fault-tolerant metric dimension in other fields of science and technology. For a study of other degree and distance related graph parameters, we refer to the papers [12,15,16,20,23,30,31].

In view of the above discussion, it is natural to study the fault-tolerant metric dimension of the families of graphs that are interesting from both combinatorial and geometric perspectives. In this paper, we study the fault-tolerant metric dimension problem for convex polytopes. We consider six well-known infinite families of convex polytopes to study their fault-tolerant resolvability. By using certain relations between resolving and fault-tolerant resolving sets, we show that these convex polytopes are the families of graphs with constant fault-tolerant metric dimension. By considering other families of structurally similar convex polytopes, we raised some open questions of more general nature at the end of this paper. Our results in this paper might give some ideas to answer some of these questions.

2. Preliminaries

In this section, we give some definitions and introduce certain terminologies that are subsequently used in the later sections. In this paper, we follow the standard notation of graph theory from the book by Bondy and Murty [6].

All graphs in this paper are finite, simple, undirected and connected. Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The *topological distance* or shortly *distance* between two vertices x and y, denoted by $d_\Gamma(x, y)$ is the length of a shortest path, also called *geodesic*, between x and y. When it is clear from the context that which graph Γ we mean, we will skip Γ from the notations $V(\Gamma)$, $E(\Gamma)$ and $d_\Gamma(x, y)$. A vertex $x \in \Gamma$ is said to *resolve* two vertices u and v, if $d(x, u) \neq d(x, v)$. A vertex set $R \subseteq V(\Gamma)$ is said to be *resolving* the underlying Γ, if for any two vertices $u, v \in V$ there exist a vertex x in R that resolve u and v. The minimum number of vertices in a set which resolve the whole graph Γ is called the *metric dimension* of Γ and usually denoted by $\beta(\Gamma)$. A resolving set of order $\beta(\Gamma)$ is said to be the *metric basis* of Γ. Equivalently, an ordered subset $R = (u_1, u_2, \ldots, u_\ell)$, the R-coordinates of a vertex $x \in V(\Gamma)$ are $C_R(x) = (d(x, u_1), d(x, u_2), \ldots, d(x, u_\ell))$. Then R is the resolving set of Γ if for every two vertices x, $y \in V(\Gamma)$ we have $C_R(x) \neq C_R(y)$. The graphs with metric dimension 1 were characterized by Chartrand et al. [10]. Let P_n be the path graphs on n vertices.

Theorem 1. *[10] A graph Γ has a metric dimension 1 if and only if $\Gamma \cong P_n$.*

Chartrand et al. also considered graphs with metric dimension 2. They did not characterize all graphs with metric dimension 2 but they investigated some properties of such graphs.

Theorem 2. *[10] A graph Γ with metric dimension 2 can have neither K_5 nor $K_{3,3}$ as a subgraph.*

A resolving set R is said to be *fault-tolerant*, if $R \backslash u$ is also a resolving set for every $x \in R$. In a similar fashion to the metric dimension, the *fault-tolerant metric dimension* is the minimum cardinality of a fault-tolerant resolving set of Γ. The fault-tolerant metric dimension of graph Γ is denoted by $\beta'(\Gamma)$. A fault-tolerant resolving set of order $\beta'(\Gamma)$ is called a *fault-tolerant metric basis* of Γ.

For a family of graphs $\mathcal{G}$, we say that $\mathcal{G}$ is a family of graphs with a constant metric dimension (resp. fault-tolerant metric dimension) if for some $\Gamma \in \mathcal{G}$, $\beta(\Gamma)$ (resp. $\beta'(\Gamma)$) it does not depend on the choice of Γ. For instance, in view of the Theorems 1, the path graph P_n is a family of graphs with a constant metric dimension.

The unique fault-tolerant metric basis of P_n is formed by the two endpoints. So it is easy to see that for the path P_n on $n \geq 2$ vertices we have $\beta(P_n) = 1$ and $\beta'(P_n) = 2$. Now we present another example to explain the concepts of the metric dimension and the fault-tolerant metric dimension of graphs. Consider the tree T in Fig. 1. It can be seen below that $\beta'(T) = 10$ and $S = (1, 2, 3, 4, \ldots, 10)$ is a metric basis. For example, we have

$$C_S(x) = (11, 11, 11, 11, 10, 10, 10, 10, 1, 4),$$

$$C_S(y) = (11, 11, 11, 11, 10, 10, 10, 10, 3, 4),$$

$$C_S(v) = (8, 8, 8, 8, 3, 3, 7, 7, 8, 9),$$

$$C_S(t) = (8, 8, 8, 8, 8, 1, 3, 7, 7, 8, 9).$$

Note that from above we can see that the only vertex in S that resolves x and y is vertex 9. Likewise, only vertex 5 resolves vertices v and t. We see that $\beta'(T) = 14$ and $S \cup \{y, v, r, s\}$ is one example of fault-tolerant metric basis of T.

It is natural to investigate a relation between metric dimension and fault-tolerant metric dimension of graphs. By definition of fault-tolerant metric dimension one can observe the following natural relation

$$\beta'(\Gamma) \geq \beta(\Gamma) + 1, \tag{2.1}$$

with equality holds, if Γ is a n-cycle C_n for $n \geq 3$. We know that $\beta(C_n) = 2$, this implies that $\beta'(C_n) \geq 3$. Javaid et al. [22] showed that $\beta'(C(n)) \leq 3$ thus we obtain that $\beta'(C_n) = 3$. In a similar fashion, we obtain that the fault-tolerant metric dimension of the complete graph K_n is n.

In view of this, Hernando et al. [17] proved the following relation between the parameters $\beta(\Gamma)$ and $\beta'(\Gamma)$.