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Sharp conditions for the existence of a stationary distribution in one classical stochastic chemostat

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ABSTRACT

This paper studies the asymptotic behaviors of one classical chemostat model in a stochastic environment. Based on the Feller property, sharp conditions are derived for the existence of a stationary distribution by using the mutually exclusive possibilities known in [11, 12] (See Lemma 2.4 for details), which closes the gap left by the Lyapunov function. Further, we obtain a sufficient condition for the extinction of the organism based on two noise-induced parameters: an analogue of the feed concentration S^* and the break-even concentration λ. Results indicate that both noises have negative effects on persistence of the microorganism.

1. Introduction

Let $S(t)$ and $x(t)$ be concentrations of the nutrient and the microorganisms at time t respectively, S^0 be the input rate of the nutrient, D be the washout rates for S, D_1 be the removal rate combining the dilution rate of the chemostat and the death rate of the microorganism x, and $\frac{\mu S}{k+S}$ be the growth rate function. Then, the chemostat (a basic piece of laboratory apparatus for the continuous culture of microorganisms) is classically represented as a system of ordinary differential equations taking the form (See [1,2])

$$dS(t) = \left[D\big(S^0 - S(t)\big) - \frac{\mu S(t)x(t)}{k+S(t)}\right]dt,\ dx(t) = \left(\frac{\mu S(t)}{k+S(t)} - D_1\right)x(t)dt. \tag{1}$$

Break-even concentration $\lambda^d = \frac{kD_1}{\mu - D_1}$ is an important parameter for analyzing (1). In detail, if $\lambda^d \geq S^0$, the only washout equilibrium $E^0 = (S^0, 0)$ is globally asymptotically stable; if $\lambda^d < S^0$, the positive equilibrium $E^* = (\lambda^d, \frac{D(k+\lambda^d)(S^0-\lambda^d)}{\mu\lambda^d})$ exists and is also globally asymptotically stable, see [3,4] for details.

Due to the existence of random effects almost everywhere in the reality, it is natural to address what happens when the stochastic perturbation is taken into account. For instance, Campillo et al. [5] established a set of stochastic chemostat models that are valid at different scales and expound the mechanism to switch from one model to another. Imhof and Walcher [6] introduced a rigorous method to get the stochastic chemostat model by defining a discrete time Markov chain and proving its convergence. The above method has also been used in [7,8] to establish the studied stochastic models. By using the method in [6], a stochastic chemostat model with the classical Monod growth rate function can be written as

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follows

$$\begin{cases} dS(t) = & \left[D\left(S^0 - S(t)\right) - \frac{\mu S(t)x(t)}{k+S(t)}\right]dt + \sigma_0 S(t)dB_0(t), \\ dx(t) = & \left(\frac{\mu S(t)}{k+S(t)} - D_1\right)x(t)dt + \sigma_1 x(t)dB_1(t). \end{cases} \tag{2}$$

$B_0(t)$ and $B_1(t)$ are two independent Brownian motions. The parameters in model (2) are all positive. Then for any initial value $(S(0), x(0)) \in R_+^2$, model (2) has a uniquely global solution $(S(t), x(t)) \in R_+^2$ *a.s.* for all t $\geq$ 0 (See [6, Proposition 6]).

Our goal is to establish an almost perfect condition for existence of the stationary distribution for (2). Further, we derive the sufficient condition for extinction of the microorganism.

Remark 1. Wang and Jiang [8] have studied the stationary distribution for the stochastic chemostat model with general response functions. Except for the linear growth rate, the actual growth rate function will be less than its value at S^0 in time average subject to the effects of noises. However the averaged growth rate function can not be applied in the Lyapunov method [8, Lemma 2.1], and the sufficient condition has to be derived under extra conditions, and is not the optimal condition. Different from the above, in this paper we employe the Feller property and mutually exclusive possibilities to derive the condition for existence of the stationary distribution, which can close the gap left by using the Lyapunov method. The method of this paper can be extended to study the stochastic chemostat models with general response functions.

2. Conditions for the stationary distribution

To begin with, let's prepare some basic results.

Lemma 2.1 [9, Lemma 4.9]**.** *For any initial value $\varphi(0) \in R_+$, consider the equation*

$$d\varphi(t) = D\left(S^0 - \varphi(t)\right)dt + \sigma_0\varphi(t)dB_0(t), \ t > 0. \tag{3}$$

Then (3) has the stationary distribution

$$p(x) = \frac{b^a}{\Gamma(a)} x^{-(a+1)} e^{-\frac{b}{x}} \text{ for } x \in R_+ \text{ with } a = \frac{2}{\sigma_0^2}\left(D + \frac{\sigma_0^2}{2}\right), \ b = \frac{2DS^0}{\sigma_0^2} \text{ and } \Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt.$$

Lemma 2.2. *Let $\varphi(t)$ be solution of (3), then* $\lim\limits_{t\to\infty} \frac{\ln(k+\varphi(t))}{t} = 0$ *a.s.*

Proof. Let $\xi(t)$ be solution of $d\xi(t) = -D\xi(t)dt + \sigma_0\xi(t)dB_0(t)$ with $\xi(0) = 1$. By using Itô formula to $\ln\xi(t)$ and then taking integrations, $\xi(t) = e^{-(D+\frac{\sigma_0^2}{2})t+\sigma_0 B_0(t)}$. Then, by applying the variation-of-constants formula, it yields $\varphi(t) = \varphi(0)e^{-(D+\frac{\sigma_0^2}{2})t+\sigma_0 B_0(t)} + DS^0\int_0^t e^{-(D+\frac{\sigma_0^2}{2})(t-s)+\sigma_0(B_0(t)-B_0(s))}ds$. Then $\varphi(t) \leq \varphi(0)e^{\sigma_0 B_0(t)} + \frac{DS^0}{D+\frac{\sigma_0^2}{2}} e^{\sigma_0 \sup\limits_{0\leq s\leq t} 2|B_0(s)|}$. Noting $\lim\limits_{t\to\infty} \frac{\sup\limits_{0\leq s\leq t}|B_0(s)|}{t} = 0$, we get the desired result. □

Remark 2. Similar to proof of Lemma 2.2, $\lim\limits_{t\to\infty} \frac{1}{t}(\ln(k + S(t) + x(t))) = 0$ *a.s.*

Remark 3. Note the positivity of ($S(t)$, $x(t)$) and $\varphi(t)$, then $S(t) \leq \varphi(t)$ a.s. holds due to the stochastic comparison theorem.

Lemma 2.3. *Let* ($S(t)$, $x(t)$) *be solution of (2) with initial value* $(S(0), x(0)) \in R_+^2$, *then for any* $p \in [1, 1 + \frac{2\min\{D, D_1\}}{\max\{\sigma_0^2, \sigma_1^2\}}]$, *it holds that*

$$E[S(t) + x(t)]^p \leq \frac{2K(p)}{\gamma(p)} + [S(0) + x(0)]^p e^{-\frac{p\gamma(p)}{2}t} \text{ and } \limsup_{t\to\infty} \frac{1}{t}\int_0^t E[S(s) + x(s)]^p ds \leq \frac{2K(p)}{\gamma(p)},$$

where $K(p) = \sup\limits_{x>0}\{DS^0x^{p-1} - \frac{\gamma(p)}{2}x^p\}$ *and* $\gamma(p) = [\min\{D, D_1\} - \frac{(p-1)}{2}\max\{\sigma_0^2, \sigma_1^2\}]$.

Proof. Applying Itô formula to model (2) yields

$$\begin{aligned} d[S(t) + x(t)]^p &= p\left\{[S(t) + x(t)]^{p-1}\left[DS^0 - DS(t) - D_1x(t)\right] + \frac{(p-1)}{2}[S(t) + x(t)]^{p-2}\left(\sigma_0^2S^2(t) + \sigma_1^2x^2(t)\right)\right\}dt \\ &\quad + p[S(t) + x(t)]^{p-1}(\sigma_0 S(t)dB_0(t) + \sigma_1 x(t)dB_1(t)) \\ &\leq p\left\{DS^0[S(t) + x(t)]^{p-1} - \left[\min\{D, D_1\} - \frac{(p-1)}{2}\max\left\{\sigma_0^2, \sigma_1^2\right\}\right][S(t) + x(t)]^p\right\}dt \\ &\quad + p[S(t) + x(t)]^{p-1}(\sigma_0 S(t)dB_0(t) + \sigma_1 x(t)dB_1(t)) \\ &\leq p\left\{K(p) - \frac{\gamma(p)}{2}[S(t) + x(t)]^p\right\}dt + p[S(t) + x(t)]^{p-1}(\sigma_0 S(t)dB_0(t) + \sigma_1 x(t)dB_1(t)) \end{aligned}$$