Applied Mathematics and Computation 339 (2018) 206–219

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

The adaptive Ciarlet–Raviart mixed method for biharmonic problems with simply supported boundary condition☆

Yidu Yang*, Hai Bi, Yu Zhang

School of Mathematical Sciences, Guizhou Normal University, Guiyang 550001, China

ARTICLE INFO

Keywords:
Ciarlet–Raviart mixed method
A priori/a posteriori error estimates
Biharmonic equation
Biharmonic eigenvalue problem
Simply supported boundary condition

ABSTRACT

In this paper, we study the adaptive fashion of the Ciarlet–Raviart mixed method for biharmonic equation/eigenvalue problem with simply supported boundary condition in $\mathbb{R}^d$. We propose an a posteriori error indicator of the Ciarlet–Raviart approximate solution for the biharmonic equation and an a posteriori error indicator of the Ciarlet–Raviart approximate eigenfuction, and prove the reliability and efficiency of the indicators. We also give an a posteriori error indicator for the approximate eigenvalue and prove its reliability. We design an adaptive Ciarlet–Raviart mixed method with piecewise polynomials of degree less than or equal to m, and numerical experiments show that numerical eigenvalues obtained by the method can achieve the optimal convergence order $O(dof^{-\frac{2m}{d}})(d=2, m=2,3; d=3, m=3)$.

1. Introduction

Biharmonic problems, including biharmonic equations and biharmonic eigenvalue problems, are fundamental and important in mathematical and physical sciences. So, there have many researches on the traditional conforming, non-conforming and mixed finite element methods (see, e.g., [1–7]), and the C^0 interior penalty Galerkin method (C^0IPG method) developed in the last decade (see [8]) for them.

The Ciarlet–Raviart mixed method (C–R mixed method) is popular for biharmonic equations (see, e.g., [4,9–14]), and has been applied successfully to biharmonic eigenvalue problems (see Section 11.3 in [1] and references therein), the quad-curl eigenvalue problem (see [15]) and the Helmholtz transmission eigenvalue problem (see [16]), etc. This paper aims to study the a posteriori error estimate and the adaptive fashion of the C–R mixed method for biharmonic problems with simply supported boundary condition in $\mathbb{R}^d$, and the features are as follows:

(1) Adaptive finite element methods are the mainstream in current scientific computing. Some good results have been systemically summarized in the treatises [6,17,18]. Many researchers studied the a posteriori error and adaptive finite element methods for biharmonic problems, e.g., the conforming finite element [6,18], the nonconforming elements [6,19–23], the C^0IPG method [8,24], and so on. However, we do not find any report on a posteriori error estimates and adaptive algorithms of the C–R mixed method for the biharmonic equation/eigenvalue problem. Since the C–R mixed method transforms the biharmonic equation with simply supported boundary condition into a system of two Laplace equations, using the a posteriori error analysis method for the conforming finite element approximation of Laplace

☆ Project supported by the National Natural Science Foundation of China (Grant No. 11561014).

* Corresponding author.
E-mail addresses: ydyang@gznu.edu.cn (Y. Yang), bihaimath@gznu.edu.cn (H. Bi), zhang_hello_hi@126.com (Y. Zhang).

https://doi.org/10.1016/j.amc.2018.07.014
0096-3003/

equation we propose an a posteriori error indicator of the C–R approximate solution for the biharmonic equation and an a posteriori error indicator of the C–R approximate eigenfuction, and prove their reliability and efficiency. By the basic relation between the error of the approximate eigenvalue and the error of the approximate eigenfuction (see Theorem 2.3), we also give an a posteriori error indicator for the approximate eigenvalue and prove its reliability. By the given error indicators we design an adaptive C–R mixed method with piecewise polynomials of degree less than or equal to m. Numerical experiments show that for the biharmonic eigenvalues which corresponding eigenfunctions have local low smoothness, numerical eigenvalues obtained by the method can achieve the optimal convergence order $O(dof^{-\frac{2m}{d}})$ $(d=2,m=2,3; d=3,m=3)$.

(2) For biharmonic problems in three dimensions, it is difficult to implement $H^2(\Omega)$ conforming method using tetrahedron meshes. Many nonconforming elements such as the Morley element and the Zienkiewicz element have their three dimensional versions at present (e.g., see [6,20]). However these nonconforming elements only use low order polynomials and hence are not efficient for capturing solutions. As for the C^0IPG method, to the best of our knowledge, there is no report on the a posteriori error estimates in three dimensions. Whereas the C–R mixed method is simple and easy to implement, moreover, it can use high order polynomials and hence can capture smooth solutions efficiently, and can also capture solutions with local low smoothness efficiently in adaptive fashion.

(3) The C–R mixed method is efficient. In Section 4, numerical results show that we can get the same accurate numerical eigenvalues by the the C–R mixed method with less degrees of freedoms comparing with the quadratic C^0IPG method, the Argyris method, and the Morley non-conforming element method (see Remark 4.1).

It is well known the main weakness of the C–R mixed method lies in that it may obtain spurious solutions for non-convex polygonal domains (see Remark 2.1). However, due to the above three features, it is still meaningful to study the adaptive C–R mixed method for biharmonic problems with simply supported boundary condition.

Throughout this paper, the letter C (with or without subscripts) denotes a positive constant independent of mesh size h, which may not be the same constant in different places. For simplicity, we use the symbol $a \lesssim b$ to mean that $a \le Cb$.

2. Preliminary

Consider the following biharmonic equation and eigenvalue problem:

$$\Delta^2\phi = \varrho f \quad in\ \Omega, \qquad \Delta\phi = \phi = 0 \quad on\ \partial\Omega; \tag{2.1}$$

$$\Delta^2 u = \lambda\varrho u \quad in\ \Omega, \qquad \Delta u = u = 0, \ \ on\ \partial\Omega, \tag{2.2}$$

where $\Omega \subset \mathbb{R}^d$ $(d \ge 2)$ is a bounded polygonal domain with boundary $\partial\Omega$, ϱ is an appropriate smooth function and has a uniform positive lower bound.

When $\Omega \subset \mathbb{R}^2$, (2.1) and (2.2) are related to plate bending and plate vibration with simply supported boundary condition, respectively.

Let $H^r(\Omega)$ denote the usual Sobolev space of real order r with norm $\|\cdot\|_r$. $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, $H^0(\Omega) = L^2(\Omega)$. Denote

$$a(v,\psi) = \int_\Omega v\psi \mathrm{d}x, \quad b(\psi, v) = \int_\Omega \nabla\psi \cdot \nabla v \mathrm{d}x, \quad (\psi, v)_\varrho = \int_\Omega \varrho\psi v \mathrm{d}x, \quad \widehat{v} = \varrho^{-1}v.$$

It's obvious that $(\cdot,\ \cdot)_\varrho$ is an inner product on $L^2(\Omega)$ and $(\cdot,\cdot)_\varrho^{\frac{1}{2}}$ is a norm equivalent to $\|\cdot\|_0$, and $\|\widehat{v}\|_0 \lesssim \|v\|_0 \lesssim \|\widehat{v}\|_0$.

Introduce the auxiliary variable $\rho = -\Delta\phi$, then we arrive at the mixed variational form of (2.1): find $(\rho,\phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$-a(\rho,\psi) + b(\psi,\phi) = 0, \qquad \forall\ \psi \in H_0^1(\Omega); \tag{2.3}$$

$$b(\rho, v) = (f, v)_\varrho, \qquad \forall\ v \in H_0^1(\Omega). \tag{2.4}$$

Let $\sigma = -\Delta u$, then (2.2) can be written as a mixed variational form: find $(\lambda,\sigma,u) \in \mathbb{R} \times H_0^1(\Omega) \times H_0^1(\Omega)$, $(\sigma,u) \ne (0,0)$, such that

$$-a(\sigma,\psi) + b(\psi,u) = 0, \qquad \forall\ \psi \in H_0^1(\Omega); \tag{2.5}$$

$$b(\sigma, v) = \lambda(u, v)_\varrho, \qquad \forall\ v \in H_0^1(\Omega). \tag{2.6}$$

Remark 2.1. When $\Omega \subset \mathbb{R}^2$, the solution of the Ciarlet–Raviart system (2.3) and (2.4) coincides with the solution of the simply-supported biharmonic problem (2.1) if and only if Ω is convex, and the solution of (2.3) and (2.4) may be spurious solution for (2.1) when Ω is nonconvex (see page 83 in [8], Section 5.9 in [3] and [12,25]). In general, when $\Omega \subset \mathbb{R}^d (d \ge 2)$ and $\phi \in H^3(\Omega)$, (2.1) can be written as the equivalent two Laplace equations