Applied Mathematics and Computation 339 (2018) 220–230

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A note on Katugampola fractional calculus and fractal dimensions

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ARTICLE INFO

MSC:
28A80
28A78
26A33
26A45

Keywords:
Katugampola fractional calculus
Bounded variation
Unbounded variation point
Hausdorff dimension
Box dimension

ABSTRACT

The goal of this paper is to study the Katugampola fractional integral of a continuous function of bounded variation defined on a closed bounded interval. We note that the Katugampola fractional integral of a function shares some analytical properties such as boundedness, continuity and bounded variation of the function defining it. Consequently, we deduce that fractal dimensions – Minkowski dimension and Hausdorff dimension – of the graph of the Katugampola fractional integral of a continuous function of bounded variation are one. A natural question then arises is whether there exists a continuous function which is not of bounded variation with its graph having fractal dimensions one. In the last part of the article, we construct a continuous function, which is not of bounded variation and for which the graph has fractal dimensions one. The construction enunciated herein includes previous constructions found in the recent literature as special cases. The article also hints at an upper bound for the upper box dimension of the graph of the Katugampola fractional derivative of a continuously differentiable function.

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1. Introduction

The concept of differentiation and integration of non-integer order is nearly as old as classical calculus and it is the subject of study in Fractional Calculus (FC). There are several books devoted to FC and its applications, among which the encyclopedic treatise by Samko et al. [24] seems to be the most prominent. A survey of many emerging applications of FC in diverse areas of science and engineering can be found in [21]. In Refs. [18,19,22,23] attempts have been made to provide a geometric and physical interpretation of fractional integration and differentiation, which is comparable with the simple interpretations of their integer-order counterparts. It is worth to mention that in these attempts, much effort has been devoted to relate FC to Fractal Geometry. Given the inherent naturalness of the subject, applications in science and engineering including fractals, and the variety of questions it generates, it is not surprising that FC continues to be an active area of research.

In FC, fractional derivatives are generally defined through fractional integrals; see, for instance, [20,24]. There are various formulations for the notion of fractional integral and a convenient approach has to be chosen depending on the modeling problem at hand. Recently, Katugampola [11] produced a fractional integral that generalizes both the Riemann–Liouville and Hadamard fractional integrals – the two competing forms of fractional integrals that have been studied extensively for their applications. According to the literature, the newly defined fractional integral is known as the Katugampola fractional

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https://doi.org/10.1016/j.amc.2018.07.035
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integral [26], and in this paper we shall continue to use the same terminology. As a natural follow-up, the Katugampola fractional derivative is defined in [12] and studied in detail in references [5–7,29]. These new approaches to fractional integral and derivative have found applications in fields such as probability theory [1], numerical analysis [3], inequalities [28] and variational principle [4].

Among the diverse issues associated to FC, an important research problem that gained recent interest is to estimate fractal dimensions (such as the box dimension and Hausdorff dimension) of the graph of the fractional integral of a function. The definitions and basic results on the aforementioned approaches to fractal dimensions are assembled in Section 2. Liang [13] studied the box dimension of the graph of the Riemann–Liouville fractional integral of a continuous function of bounded variation on a closed bounded interval. The box dimension of the graph of a continuous function which is not of bounded variation can be consulted in [14,15,17,25]. More recently, the box dimension of the graph of the Hadamard fractional integral of a continuous function of bounded variation and not of bounded variation is investigated in [27].

The first part of the current article contributes in establishing fractal dimensions of the graph of the fractional integral of a continuous function in the context of the Katugampola fractional integral and hence may be viewed as a sequel to [13,27]. In Section 3, we shall note that the Katugampola fractional integral of a function preserves the basic analytical properties such as boundedness, continuity, and bounded variation of the function that defines the integral. Consequently, we deduce that fractal dimensions of the graph of the Katugampola fractional integral of a continuous function of bounded variation on a closed bounded interval in $\mathbb{R}$ are one. We should admit that the influence of Ref. [13] on this part of our research reported here goes further. Besides providing us with the motivating question, it also offered us an array of basic tools which we have modified and adapted.

Here and throughout this paper, let us refer to a function which is not of bounded variation as a function of unbounded variation. With a slight abuse of terminology, we shall refer to the dimension of the graph of a function as the dimension of the function itself. Note that according to the traditional definition, the property of bounded variation of a function is global in nature, since it is defined over a set rather than at a point in the domain. Targeting at the classification of one-dimensional continuous functions, Liang [14] expanded this notion and distinguished a bounded variation point and an unbounded variation point for a function; see Section 2 for precise definitions and relevant notation. Let us mention that fractal dimensions of the graph of a continuous function of bounded variation are one [13]. On the other hand, the most popular example of a continuous nowhere differentiable function by Weierstrass is of unbounded variation and its graph has fractal dimensions larger than one, see [10]. A natural question then arises as to whether there exists a one-dimensional continuous function of unbounded variation; see also [2]. One can go further and ask if there exists a one-dimensional continuous function with a finite, countable or uncountable number of unbounded variation points. More recently, progress has been made in the construction of specific examples of continuous one-dimensional functions of unbounded variation, details can be consulted in [14–16,25,27]. The second part of the paper is in this vein. To be precise, in Section 4, we present a new method for the construction of a continuous function of unbounded variation from a prescribed continuous function, which we call the generating function. For the constructed function, the cardinality of the set of points of unbounded variation and fractal dimensions of the graph depend on the choice of the generating function. Furthermore, we believe that our construction is conceptually more general and includes some of the constructions of this kind available in the recent literature as special cases.

2. Preliminaries

In this section we lay out the background material. The reader, if so inclined, may consult [8,9,11,12] for details.

2.1. Fractal dimensions

For a non-empty subset U of $\mathbb{R}^n$, the diameter is defined as

$$|U| = \sup\{\|x-y\| : x, y \in U\}.$$

If $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most δ that cover $F \subseteq \mathbb{R}^n$, then we say that $\{U_i\}$ is a δ-cover of F. Let s be a non-negative real number and $\delta > 0$. Define

$$H^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \quad \delta-\text{cover} \quad \text{of } F\right\}.$$

Definition 2.1. The s-dimensional Hausdorff measure of F is defined by $H^s(F) = \lim_{\delta\to 0} H^s_\delta(F)$.

Definition 2.2. Let $F \subseteq \mathbb{R}^n$ and $s \geq 0$. The Hausdorff dimension of F is defined as

$$\dim_H(F) = \inf\left\{s : H^s(F) = 0\right\} = \sup\{s : H^s(F) = \infty\}.$$

Remark 2.3. If $s = \dim_H(F)$, then $H^s(F)$ may be zero or infinite, or may satisfy $0 < H^s(F) < \infty$. A Borel set satisfying this last condition is called an s-set.

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