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Graphs preserving Wiener index upon vertex removal

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ABSTRACT

The Wiener index $W(G)$ of a connected graph G is defined as the sum of distances between all pairs of vertices in G. In 1991, Šoltés posed the problem of finding all graphs G such that the equality $W(G) = W(G - v)$ holds for all their vertices v. Up to now, the only known graph with this property is the cycle C_{11}. Our main object of study is a relaxed version of this problem: Find graphs for which Wiener index does not change when a particular vertex v is removed. In an earlier paper we have shown that there are infinitely many graphs with the vertex v of degree 2 satisfying this property. In this paper we focus on removing a higher degree vertex and we show that for any $k \geq 3$ there are infinitely many graphs with a vertex v of degree k satisfying $W(G) = W(G - v)$. In addition, we solve an analogous problem if the degree of v is $n - 1$ or $n - 2$. Furthermore, we prove that dense graphs cannot be a solutions of Šoltes's problem. We conclude that the relaxed version of Šoltés's problem is rich with a solutions and we hope that this can provide an insight into the original problem of Šoltés.

1. Introduction

Average distance is one of the three most robust measures of network topology, along with its clustering coefficient and its degree distribution. Nowadays it has been frequently used in sociometry and the theory of social networks [4]. Wiener index, defined as the sum of distances between all (unordered) pairs of vertices in a graph, besides its crucial role in the calculation of average distance, is the most famous topological index in mathematical chemistry. It is named after Wiener [12], who introduced it in 1947 for the purpose of determining boiling points of alkanes. Since then Wiener index has become one of the most frequently used topological indices in chemistry as molecules are usually modelled by undirected graphs. Other applications of this graph invariant can be found in crystallography, communication theory and facility location. Wiener index has also been studied in pure mathematics under various names: the gross status, the distance of a graph, the transmission of a graph etc. It seems that the first mathematical paper on Wiener index was published in 1976 [3]. Since then, a lot of mathematicians have studied this quantity very extensively. A great deal of knowledge on Wiener index is accumulated in survey papers [2,6–9,13].

Let G be a finite, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices in G is usually denoted by $n(G)$. For $u, v \in V(G)$ the *distance* $d_G(u, v)$ between vertices u and v is defined as the number of edges

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on a shortest path connecting these vertices in G. The distance, or transmission, $t_G(v)$ of a vertex $v \in V(G)$ is the sum of distances between v and all other vertices of G. By $G-v$ we denote a graph obtained from G when v and all edges incident with v are deleted.

The *Wiener index* $W(G)$ of a connected graph G is defined as:

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) = \frac{1}{2}\sum_{v\in V(G)} t_G(v). \tag{1}$$

Although many papers have been devoted to Wiener index, there are still some interesting open problems concerning this quantity, and they are still quite challenging because they defy all attempts to solve them for a long time. Therefore, Wiener index is still a very popular subject of study in pure and applied mathematics.

One particular problem concerning Wiener index was posed by Šoltés [11] in 1991:

Problem 1. Find all graphs G in which the equality $W(G) = W(G-v)$ holds for all $v \in V(G)$.

He managed to discover only one such graph, namely the cycle C_{11}. We will conveniently call graphs satisfying $W(G) = W(G-v)$ for all v as *Šoltés's graphs*. So far it is not known whether there are other solutions besides C_{11} so Šoltés's problem is still on the list of unsolved problems in graph theory. In a previous paper [5] we studied a relaxed version of Problem 1.

Problem 2. Find all graphs G for which the equality $W(G) = W(G-v)$ holds for at least one vertex v.

We have shown that the class of graphs for which Wiener index does not change when a particular vertex is removed is rich even if we focus only on unicyclic graphs. We constructed an infinite class of graphs G with a vertex v of degree 2 such that $W(G) = W(G-v)$.

In this paper we extend our research to graphs in which vertex v is of arbitrary degree. For $k \geq 3$ we show that there are infinitely many graphs G with a vertex v of degree k for which $W(G) = W(G-v)$. Moreover, we prove the existence of such graphs when the degree is $n-1$ or $n-2$. Finally, we show that dense graphs cannot provide a solution of Problem 2, and therefore, they cannot be Šoltés's graphs.

2. Preliminaries

Let G be a connected graph. By $d_G(v)$ we denote the degree of vertex v. The index will be omitted when no confusion is likely. A *pendant vertex* is a vertex of degree one and a *pendant edge* is an edge incident with a pendant vertex. A *diameter* diam(G) of a graph G is the greatest distance in G. By K_n we denote an n-vertex *complete graph*, by P_n n-vertex *path* and by S_n an n-vertex *star*. For more definitions and terminologies in graph theory, see [1].

Polansky et al. [10] studied behavior of Wiener index due to some graph operations and they proved the following.

Lemma 3. *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, respectively. Further, let $u \in V(G_1)$, $z \in V(G_2)$ and let H arises from G_1 and G_2 by identifying u and z. Then*

$$W(H) = W(G_1) + W(G_2) + (n_1-1)t_{G_2}(z) + (n_2-1)t_{G_1}(u). \tag{2}$$

We start with an observation that will be used later.

Proposition 4. *Let $n \geq 2$. Further, let t satisfies $n-1 \leq t \leq \binom{n}{2}$. Then there exists an n-vertex tree T containing a vertex v such that $t_T(v) = t$.*

Proof. We use induction on n. For $n = 2$ we have $t = 1$. Such a tree is unique, namely P_2. Hence, suppose that the statement of proposition is true for $n = l$ and in what follows assume that $n = l+1$. Then $l \leq t \leq \binom{l+1}{2}$.

Suppose first that $l \leq t \leq \binom{l}{2} + 1$. By the induction hypothesis, there exists a tree T with l vertices and a vertex v such that $t_T(v) = t-1$. By adding one pendant vertex to v we get a tree T' with $l+1$ vertices such that $t_{T'}(v) = t$.

So suppose that $\binom{l}{2} + 2 \leq t \leq \binom{l}{2} + l = \binom{l+1}{2}$. Take a path P_l and v as the one of its two end-vertices. Since $t_{P_l}(v) = \binom{l}{2}$, by introducing a pendant vertex in P_l that is on distance s from v, we obtain T' with $l+1$ vertices and $t_{T'}(v) = \binom{l}{2} + s = t$. □

Note that the extreme values are attained for S_n and P_n.

3. Vertices of prescribed degree

Our first observation shows that if a vertex v has degree 1 in G, then $W(G) > W(G-v)$. If G is a connected graph, $v \in V(G)$ is a pendant vertex and uv is the pendant edge in G, then for all $x \in V(G)$, $x \neq v$ we have

$$d_G(v,x) = d_{G-v}(u,x) + 1.$$

Hence $W(G) = W(G-v) + t_{G-v}(u) + n(G-v)$.