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Reversibility in polynomial systems of ODE's

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ABSTRACT

For a given family of real planar polynomial systems of ordinary differential equations depending on parameters, we consider the problem of how to find the systems in the family which become time-reversible after some affine transformation. We first propose a general computational approach to solve this problem, and then demonstrate its usage for the case of the family of quadratic systems.

1. Introduction

One of important problems arising in the investigation of the qualitative behavior of dynamical systems is determining whether a given system admits some kind of symmetry. In studies of dynamical systems described by autonomous polynomial systems of ordinary differential equations, we deal mainly with two kinds of symmetries: The rotational symmetry ($\mathbb{Z}_q$-symmetry) and the time-reversible (involutive) symmetry.

$\mathbb{Z}_q$-invariant systems arise frequently in works related to the second part of Hilbert's 16th problem, since, due to the existence of such symmetry, it is possible to construct polynomial systems with many limit cycles (see e.g. [6,9,13–15,20] and references therein).

The existence of time-reversible symmetry in a polynomial system is related closely to the integrability of the system. For example, consider the real system

$$\dot{u} = -v - vU(u, v^2), \qquad \dot{v} = u + V(u, v^2), \tag{1.1}$$

where U is an analytic function without constant term, and V is an analytic function whose series expansion at (0,0) starts with terms of order at least two. Clearly, the origin of system (1.1) is either a focus or a center. It is seen easily seen that the phase portrait of system (1.1) remains unchanged after reflection with respect to the Ou axis, and reversion of the sense of every orbit (the reversal of time). Hence, the origin of (1.1) is a center and, by the Poincaré–Lyapunov theorem, the system admits a local analytical integral. In general, it is said that a smooth differential system

$$\dot{x} = F(x), \qquad x \in \mathbb{R}^n, \tag{1.2}$$

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where $F(x)$ is an n-dimensional real vector valued function, is *time-reversible* if there exists a diffeomorphism A satisfying

$$A \circ A = Id \tag{1.3}$$

such that

$$\frac{d(Ax)}{dt} = -F(Ax).$$

The map A satisfying (1.3) is called an *involution*. In recent years, different aspects of involutive symmetry of differential systems have been investigated extensively by many authors, see e.g. [5,8,11,12,17,19,21] and the references therein. In particular, it was shown in [16] that, for time-reversible systems

$$\dot{x} = F_0 x + F_1(x), \tag{1.4}$$

where F_0 is a matrix and $F_1(x)$ is a series convergent in a neighborhood of the origin which expansion starts with at least quadratic terms, the number of first integrals of system (1.4) is the same as the number of first integrals of the linear system $\dot{x} = F_0 x$.

In the present paper, we limit our consideration to the case of two-dimensional polynomial systems of ordinary differential equations and the linear non-scalar orthogonal involutions A, i.e. linear maps representing orthogonal reflections over a line passing through the origin. The main problem of our study is the following one:

Given a system (1.2), where $F(x, y) = (F_1(x, y), F_2(x, y))$, F_1 and F_2 are polynomials depending on parameters, how to find the set in the space of parameters corresponding to the systems which become time-reversible after some affine transformation.

Following [1,18], we write polynomial systems (1.2) on $\mathbb{R}^2$ in the form

$$\begin{aligned} \dot{x} &= -\sum_{0\le p+q<n,\, p\ge -1,\, q\ge 0} a_{pq} x^{p+1} y^q = P(x, y), \\ \dot{y} &= \sum_{0\le p+q<n,\, p\ge -1,\, q\ge 0} b_{qp} x^q y^{p+1} = Q(x, y), \end{aligned} \tag{1.5}$$

and abbreviate by **a** the vector of parameters of the first equation, and by **b** the vector of parameters of the second equation.

In more detail, the questions of our interest are:

(i) How to find conditions on the coefficients of system (1.5), which describe those systems in the family which allow a *linear transformation* into a time-reversible system.

(ii) How to find conditions on the coefficients of system (1.5), which describe those systems in the family which allow an *affine transformation* into a time-reversible system.

Note that a similar problem as in (i) has been considered in [17,18]. However, there, only a two-parameter family of linear transformations was taken into account. Our aim here is to extend the set of transformations first to the set of general linear transformations, and then to the set of affine transformations.

We propose a computational approach to answer these questions which is based on algorithms of the elimination theory.

Our paper is organized as follows. In the next section we present some general results related to affine transformations and time-reversibility in plane systems (1.2). Based on these results, in Section 3 we describe a computational approach for solving the problems stated above, and in Section 4 apply it to study in detail time-reversibility of the quadratic system

$$\begin{aligned} \dot{x} &= -(a_{00}x + a_{-11}y + a_{10}x^2 + a_{01}xy + a_{-12}y^2), \\ \dot{y} &= b_{1,-1}x + b_{00}y + b_{2,-1}x^2 + b_{10}xy + b_{01}y^2. \end{aligned} \tag{1.6}$$

As to notations of this paper, we adopt writing the action of a linear map without parenthesis. Linear maps will be denoted by capital letters, points of $\mathbb{R}^2$ in boldface, and non-linear maps in calligraphic font. We write briefly $\mathbf{z}$ for the map $t \mapsto \mathbf{z}(t) = (x(t), y(t))$, $t \in \mathbb{R}$. The same symbol may also denote a point in the space $\mathbb{R}^2$, which will be clear from the context. We use the same letter for either a linear map or, depending on the context, its matrix in the standard basis of the underlying vector space. The symbol Id stands for both the identity map and the identity matrix.

2. Time-reversibility with respect to linear involutions

We say that a linear map $A : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a non-trivial involution if $A^2 = Id$ and $A \neq \pm Id$. If A is, additionally, orthogonal ($AA^{tr} = A^{tr}A = Id$, where A^{tr} denotes the transpose of A), it will be called a *reflection*. It is well-known that such an A is orthogonally diagonalizable with eigenvalues 1 and -1 and therefore also symmetric. So we have $A = A^{tr} = A^{-1}$. Furthermore, the line of fixed points of a reflection A is exactly its eigenspace corresponding to the eigenvalue $\lambda = 1$, and the eigenspace for $\lambda = -1$ determines the line along which the map A reflects.

Let $\mathbf{z} = \mathbf{z}(t) = (x(t), y(t)) \in \mathbb{R}^2$. The system

$$\frac{d\mathbf{z}}{dt} = \mathcal{F}(\mathbf{z}) \tag{2.1}$$