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On the lacunary sum of trinomial coefficients

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ABSTRACT

The trinomial coefficient $\binom{n}{k}_2$ is given by

$$\sum_{k=-n}^{n}\binom{n}{k}_2 x^k=(1+x+x^{-1})^n.$$

In this paper, we obtain the explicit formulas for the lacunary sum

$$\sum_{\substack{-n\le k\le n\\ k\equiv r \pmod m}}\binom{n}{k}_2.$$

For example,

$$\sum_{\substack{-n\le k\le n\\ k\equiv 1 \pmod{12}}}\binom{n}{k}_2=\frac{2^n+3^n-(-1)^n+6H_n}{12},$$

where $H_0=0$, $H_1=1$ and $H_n=2H_{n-1}+2H_{n-2}$ for $n\ge 2$.

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1. Introduction

The trinomial coefficient $\binom{n}{k}_2$ is given by

$$\sum_{k=-n}^{n}\binom{n}{k}_2 x^k=(1+x+x^{-1})^n. \tag{1.1}$$

In particular, set $\binom{n}{k}_2=0$ if $k<-n$ or $k>n$. It is easy to see that

$$\binom{n}{k}_2=\binom{n}{-k}_2$$

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and

$$\binom{n}{k}_2 = \binom{n-1}{k}_2 + \binom{n-1}{k+1}_2 + \binom{n-1}{k-1}_2.$$

The trinomial coefficients were firstly studied by Euler. Euler observed that

$$3\binom{n+1}{0}_2 - \binom{n+2}{0}_2 = F_n(F_n+1), \qquad n = 0, 1, \ldots, 7, \tag{1.2}$$

where F_n is the nth Fibonacci number given by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. However, (1.2) may fail for $n \geq 8$. In fact, (1.2) is a classical example of the second strong law of small numbers [2]. In 1990, Andrews [1] found that

$$\sum_{k=-\infty}^{\infty}\left(\binom{n}{10k}_2 - \binom{n}{10k+1}_2\right) = \frac{F_{n-1}(F_{n-1}+1)}{2} \tag{1.3}$$

for any $n \geq 1$. Clearly (1.3) implies Euler's observation, since

$$\begin{aligned} 3\binom{n+1}{0}_2 - \binom{n+2}{0}_2 &= 2\binom{n+1}{0}_2 - 2\binom{n+1}{1}_2 \\ &= 2\sum_{k=-\infty}^{\infty}\left(\binom{n+1}{10k}_2 - \binom{n+1}{10k+1}_2\right) \end{aligned}$$

whenever $0 \leq n \leq 7$. Furthermore, Andrews [1, Eq. (2.18)] completely determined the explicit formulas for

$$\sum_{k=-\infty}^{\infty}\binom{n}{10k+a}_2, \qquad a = 0, 1, \ldots, 5.$$

On the other hand, the lacunary sum of binomial coefficients

$$\sum_{\substack{0\leq k\leq n\\ k\equiv r \pmod m}}\binom{n}{k}$$

has been systematically investigated by Sun and Sun [4–9]. For example, in [7], Sun and Sun expressed the sum

$$\sum_{\substack{0\leq k\leq n\\ k\equiv r \pmod{10}}}\binom{n}{k}$$

in terms of the Fibonacci numbers $\{F_n\}$ and the Lucas numbers $\{L_n\}$, which are given by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. In general, in [9], for any $m \geq 1$ and $r \in \mathbb{Z}$, Sun obtained the explicit formula for

$$\sum_{\substack{0\leq k\leq n\\ k\equiv r \pmod m}}\binom{n}{k}.$$

Let ϕ denote the Euler totient function and let

$$\delta_m = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

Sun proved that for any $n \geq 1$,

$$\sum_{\substack{0\leq k\leq n\\ k\equiv r \pmod m}}\binom{n}{k} = \frac{1}{m}\left(\sum_{\substack{d\mid m\\ d>2}} w_{\lfloor\frac{n+1}{2}\rfloor}(n-2r, d) + 2^n + (-1)^r\delta_m\right), \tag{1.4}$$

where $\{w_n(r, d)\}$ is a linear recurrence sequence of order $\phi(d)/2$. In particular, for any odd $n \geq 1$,

$$\begin{aligned} &12\sum_{\substack{0\leq k\leq n\\ k\equiv r \pmod{12}}}\binom{n}{k} - 2^n - 1 \\ &= \begin{cases} 3^{\frac{n+1}{2}} + (-1)^{\frac{r(n-r)}{2}+\frac{n^2-1}{8}}(2^{\frac{n+1}{2}} + T_{\frac{n+1}{2}}), & \text{if } n-2r \equiv \pm 1 \pmod{12}, \\ -3 + (-1)^{\frac{r(n-r)}{2}+\frac{n^2-1}{8}}(2^{\frac{n+1}{2}} - T_{\frac{n+1}{2}} + T_{\frac{n-1}{2}}), & \text{if } n-2r \equiv \pm 3 \pmod{12}, \\ -3^{\frac{n+1}{2}} + (-1)^{\frac{r(n-r)}{2}+\frac{n^2-1}{8}}(2^{\frac{n+1}{2}} - T_{\frac{n-1}{2}}), & \text{if } n-2r \equiv \pm 5 \pmod{12}, \end{cases} \end{aligned}$$

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