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Analysis on the method of fundamental solutions for biharmonic equations

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ABSTRACT

In this paper, the error and stability analysis of the method of fundamental solution (MFS) is explored for biharmonic equations. The bounds of errors are derived for the fundamental solutions $r^2 \ln r$ in bounded simply-connected domains, and the polynomial convergence rates are obtained for certain smooth solutions. The bounds of condition number are also derived to show the exponential growth rates for disk domains. Numerical experiments are carried out to support the above analysis, which is the first time to provide the rigorous analysis of the MFS using $r^2 \ln r$ for biharmonic equations.

1. Introduction of the MFS

The method of fundamental solutions (MFS) was first used in Kupradze [14] in 1963. Since then, there have appeared numerous reports of the MFS for computation such as the reviews of the MFS in Fairweather and Karageorghis [9] and Golberg and Chen [11]. On the other hand, the Trefftz method (TM) [27] as boundary methods has been fully developed in theory and computation for several decades (see [24]), where only the particular solutions (PS) are discussed. In fact, the MFS is one of the Trefft Methods using fundamental solutions (FS). Both the MFS and the MPS can be classified into TM, and they are carried out by the collocation TM (CTM) in [24]. In contrast to the traditional methods [3,4,10,16,28], we focus on the error and stability analysis of the MFS for the biharmonic equation in this paper. In the stability analysis, the study of condition number is also crucial [17,29]. Bounds of condition number will be further investigated in this paper.

Consider the biharmonic equations and clamped boundary conditions

$$\triangle^2 u = 0, \quad \text{in } S, \tag{1.1}$$

$$u = f, \quad \text{on } \Gamma, \tag{1.2}$$

$$u_\nu = \frac{\partial u}{\partial \nu} = g, \quad \text{on } \Gamma, \tag{1.3}$$

where $\triangle = \partial^2/\partial x^2 + \partial^2/\partial y^2$, S is the bounded simply connected domain, u_ν the outward normal derivative to Γ, Γ its smooth boundary, and f, g the given functions with sufficient smoothness. Denote $r = |PQ|$, $P = \rho e^{i\theta}$, $Q = Re^{i\phi}$, and $i = \sqrt{-1}$.

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The general traditional fundamental solutions (FS) of biharmonic equations in 2D are well-known as

$$\Phi(\rho,\theta) = r^2 \ln r, \quad r = \sqrt{R^2 + \rho^2 - 2R\rho\cos(\theta - \phi)}, \tag{1.4}$$

where Q is the source point outside the domain. Denote the FS at $Q_j = Re^{i\phi_j}$ as

$$\Phi_j(\rho,\theta) = r_j^2 \ln r_j, \quad \phi_j(\rho,\theta) = \ln r_j, \tag{1.5}$$

where $r_j = |PQ_j| = \sqrt{R^2 + \rho^2 - 2R\rho\cos(\theta - \phi_j)}$. Hence we may also choose the linear combination

$$v_N = \sum_{j=1}^{N} (c_j \Phi_j(\rho,\theta) + d_j \phi_j(\rho,\theta)), \tag{1.6}$$

where $\{c_j\}$ and $\{d_j\}$ are the unknown coefficients to be determined by the boundary conditions (1.2) and (1.3). Denote V_N the set of (1.6). We may solicit the Trefftz method [24] to seek the numerical solution u_N by

$$I(u_N) = \min_{v \in V_N} I(v), \tag{1.7}$$

where

$$I(v) = \int_\Gamma (v - f)^2 + w^2 \int_\Gamma (v_\nu - g)^2, \tag{1.8}$$

and w is the weight which may be chosen as $w = 1/N$ (see [24]). The TM using the FS is exactly the very MFS (see [20,22]).

The Almansi's FS for biharmonic equations are obtained directly from the general solutions, $u(\rho,\theta) = \rho^2 v + z$, where v and z are harmonic functions, to give $\Phi^A(\rho,\theta) = \rho^2 \ln r$. We may choose the linear combination

$$v_N^A = \sum_{j=1}^{N} (c_j \Phi_j^A(\rho,\theta) + d_j \phi_j(\rho,\theta)), \tag{1.9}$$

to replace (1.6), where $\Phi_j^A(\rho,\theta) = \rho^2 \ln r_j$, and the coefficients c_j and d_j are also obtained from (1.7). Eq. (1.7) with (1.9) is called the MFS using Almansi's FS: $\rho^2 \ln r$. For biharmonic equation, the MFS using $\rho^2 \ln r$ was introduced in Karageorghis and Fairweather [9,12,13]. The analysis of the MFS for Laplace's equations can be found in [5,18,20]. In Li et al. [22], a preliminary analysis of error and stability was given for the MFS using $\rho^2 \ln r$. Note that the analysis for the MFS using $r^2 \ln r$ is more challenging and important.

This paper is organized as follows. In Sections 2 and 3, the preliminary lemmas and the main theorems are derived for error bounds, respectively. In Section 4, the bounds of condition numbers are derived for circular domains. In the last section, numerical experiments are reported to illustrate our results.

2. Preliminary lemmas

Firstly let us cite lemmas in [1,20].

Lemma 2.1. *For the periodical function $g = g(x)$ on $[0, 2\pi]$, denote the errors*

$$\Delta_N(g) = T_N(g) - \int_0^{2\pi} g(x)dx, \tag{2.1}$$

where the trapezoidal rule, $T_N(g) = h\sum_{k=0}^{N-1} g(kh)$ with $h = 2\pi/N$. Then there exist the equalities,

$$\Delta_N(\sin mx) = 0, \quad m = 1, 2, \ldots, \tag{2.2}$$

$$\Delta_N(\cos mx) = \begin{cases} 2\pi, & \text{if } m = \nu N, \quad \nu = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Lemma 2.2. *Let $g(x) \in C^\infty[0, 2\pi]$ satisfy the periodical boundary conditions,*

$$g^{(\ell)}(0) = g^{(\ell)}(2\pi), \quad \ell = 0, 1, \ldots \tag{2.4}$$

There exist the errors

$$T_N(g(x)\cos(kx)) - \int_0^{2\pi} g(x)\cos(kx)dx = \pi \sum_{\nu=1}^{\infty} \{\alpha_{\nu N - k} + \alpha_{\nu N + k}\}, \quad k = 0, 1, \ldots, N-1, \tag{2.5}$$

where

$$\alpha_\ell = \frac{1}{\pi} \int_0^{2\pi} g(x)\cos(\ell x)dx, \quad \ell = 0, 1, \ldots \tag{2.6}$$