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Normal solutions of a boundary-value problem arising in free convection boundary-layer flows in porous media

Zhongxin Zhang*

Department of Mathematics, Xiamen University, Xiamen, Fujian 361005, PR China

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ABSTRACT

This paper is concerned with normal solutions of a two-point boundary-value problem of second order which arises in the steady free convection boundary-layer flow over a vertical permeable flat plate being embedded in a saturated porous medium with both prescribed heat flux and suction rate of the plate. We use a very simple argument to prove that there exists a $\lambda_{\min} \in (1.3782407, 1.4166499)$ such that this problem has no normal solution for all $\lambda < \lambda_{\min}$, a unique normal solution for $\lambda = \lambda_{\min}$, and exactly two normal solutions $\theta_1(\eta)$ and $\theta_2(\eta)$ with $\theta_1(\eta) < \theta_2(\eta)$ on $[0, +\infty)$ for each fixed $\lambda > \lambda_{\min}$, which is applied to the reduced boundary-layer system.

1. Introduction

In the present study, we consider the following nonlinear boundary-value problem involving a parameter λ,

$$\begin{cases}\theta''(\eta) + \lambda\theta'(\eta) = -\theta^2(\eta), \ \eta \ge 0, \\ \theta'(0) = -1, \ \theta(\infty) = 0.\end{cases} \tag{1.1}$$

As indicated in Section 2, BVP (1.1) describes a steady pseudo-similarity free convection boundary-layer flow over a vertical permeable flat plate with both prescribed heat flux and suction rate of the plate which is embedded in a saturated porous medium.

Here we should point out that the following two-point boundary-value problem

$$\begin{cases}\theta''(\eta) + \lambda\theta'(\eta) = -\theta^2(\eta), \ \eta \ge 0, \\ \theta(0) = 1, \ \theta(\infty) = 0,\end{cases} \tag{1.2}$$

has been studied in [1–3], which arises in the similarity free convection boundary-layer flow adjacent to a vertical permeable surface in a porous medium with prescribed surface temperature [2]. From the numerical results in [1,2], we know that BVP (1.2) admits multiple solutions for every fixed $\lambda > \lambda^* = 1.079131$ (see [1,2]), that is to say, BVP (1.2) is not a well-posed mathematical model. From the recent paper [3], we know that BVP (1.2) with the following constraint conditions

$$\theta'(0) \le 0 \ \text{and} \ \int_0^{+\infty} \theta(s)ds < +\infty$$

* Corresponding author.
E-mail address: zzxmath@163.com

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becomes a well-posed mathematical problem, i.e., the solution of BVP (1.2) with the above constraint conditions is unique. Based on the above mentioned conclusions, we surmise that BVP (1.1) admits multiple solutions for every given sufficiently large λ, and the problem with the constraint condition $\int_0^{+\infty}\theta(s)ds < +\infty$ should be well-posed. In this paper, we are interested in positive solutions of BVP (1.1) satisfying $\int_0^{+\infty}\theta(s)ds < +\infty$.

Definition 1. A positive value function $\theta(\eta)\in C^2[0,\infty)$ is called a normal solution to BVP (1.1) if it is a positive solution to BVP (1.1) that satisfies the following constraint condition

$$\int_0^{+\infty}\theta(\eta)d\eta < +\infty.$$

Remark 1. From [3], we know that the condition $\int_0^{+\infty}\theta(\eta)d\eta$ represents the dimensionless displacement thickness of the boundary-layer problem considered in [1] and hence the constraint condition $\int_0^{+\infty}\theta(\eta)d\eta < +\infty$ is fair and reasonable. As to BVP (1.1), the condition $\int_0^{+\infty}\theta(\eta)d\eta < +\infty$ represents the the pseudo-similarity solution $f(\eta)$ is bounded, where $f'(\eta) = \theta(\eta)$, for details, see Section 2 below.

In [3], the main result about normal solutions for BVP (1.2) is that there exists a $\lambda^*\in[1, 2/\sqrt{3}]$ such that BVP (1.2) has a unique normal solution for all $\lambda\geq\lambda^*$ and no normal solution for $\lambda<\lambda^*$. It is of interest to know whether the same conclusion is valid for BVP (1.1). In this paper, our purpose is to investigate BVP (1.1) using the idea and some conclusions due to [3] and obtain the following theorem.

Theorem 1. *There exists a $\lambda_{\min}\in(1.3782407, 1.4166499)$ such that BVP (1.1) has no normal solution for all $\lambda<\lambda_{\min}$; a unique normal solution for $\lambda=\lambda_{\min}$ and exactly two normal solutions $\theta_1(\eta)$ and $\theta_2(\eta)$ for $\lambda>\lambda_{\min}$, which satisfy $\theta_1(\eta)<\theta_2(\eta)$ on $[0,+\infty)$ and there exists $\eta_*>0$ such that $\theta_2(\eta+\eta_*)=\theta_1(\eta)$ for $\eta\geq 0$.*

Remark 2. From [3] and the above theorem, we know that the conclusion about normal solutions for BVP (1.1) is very different from that for BVP (1.2).

The paper is organized as follows: In the second section, we will consider a steady pseudo-similarity free convection boundary-layer flow over a vertical permeable flat plate with both prescribed heat flux and suction rate of the plate which is embedded in a saturated porous medium and establish the BVP (1.1) utilizing the method in [2], and we give the existence result of bounded solutions for a reduced boundary-layer system. In Section 3, we will explore the properties of the normal solutions to BVP (1.1) and discuss an initial-value problem by using the results obtained in [3]. The last section contains the proof of Theorem 1 and a conjecture we propose.

2. Derivation of the model and bounded solution

Let us consider the problem of steady two-dimensional free convection around a vertical permeable flat plate with both prescribed heat flux and prescribed mass transfer rate, which is embedded in a saturated porous medium at the ambient temperature T_∞. Choose a rectangular Cartesian coordinate system with the origin fixed at the leading edge of the vertical plane surface such that the $x-$axis is directed upwards and the $y-$axis is perpendicular to the boundary plate surface and into the porous medium. We make the following assumptions (see [4–6]): (i) the porous medium is homogeneous and isotropic; (ii) the convective fluid is viscous and incompressible; (iii) all the properties of the porous medium and the fluid, such as viscosity, thermal conductivity and permeability, are taken as constants; (iv) the convective fluid and the porous medium are everywhere in local thermodynamic equilibrium; (v) the temperature of the fluid is everywhere below boiling point; (vi) the fluid velocity obeys Darcy's law and the Oberbeck–Boussinesq approximation is valid. Under these assumptions, the steady natural convection is governed by the following partial differential equations [5–11]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$u=-\frac{K}{\mu}\left(\frac{\partial p}{\partial x}+\rho g\right), \tag{2.2}$$

$$v=-\frac{K}{\mu}\frac{\partial p}{\partial y}, \tag{2.3}$$

$$\rho=\rho_\infty[1-\beta(T-T_\infty)], \tag{2.4}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right), \tag{2.5}$$

where u and v are the Darcy's velocity components in the x and y directions, ρ, μ and β are the density, viscosity and thermal expansion coefficient of the fluid, respectively. K is the permeability of the saturated porous medium, α is the equivalent thermal diffusivity, g is the gravitational acceleration, p is the pressure and T is the temperature. The subscript ∞ refers to the condition far from the plate.