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A conforming locking-free approximation for a Koiter shell

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ABSTRACT

As in the Naghdi framework, membrane locking is expected for bending-dominated Koiter shell when the thickness decreases. Inspired by Arnold and Brezzi (1997), we design a locking-free mixed finite element method for the Koiter shell. This method is implemented, in terms of the displacement variables, as the minimization of an altered energy over a conforming finite element space. We approximate the tangential displacements by continuous piecewise polynomials augmented by bubbles and the transversal displacements by the consistent *HCT* (Hsieh–Clough–Tocher) element. The membrane stresses, derived from a partial integration of the membrane energy, is approximated by discontinuous piecewise polynomials. We establish optimal error estimates independent of the thickness under some restrictions which prove that the mixed solution is locking-free. We confirm our theoretical predictions with some numerical tests, in particular, we consider a hemicylindrical shell and an hyperbolic paraboloid shell.

1. Introduction

The variational problem associated to the Koiter shell involves an internal energy as the sum $E_1 + t^{-2}E_2$ where t is the thickness and E_1, E_2 are respectively the bending and the membrane elastic energies. And as it is well-known, the convergence of the standard finite element methods deteriorates for such problems when t is too small. For the Naghdi shells, the literature is rich on the locking-free finite element methods, most of the proposed solutions are based on the introduction of mixed formulations [1,6,7,10,18,20,21]. Arnold and Brezzi proposed in [1] a suitable splitting of the membrane energy into an exactly integrated part and a reduced one at discrete level in order to weaken the membrane constraints. The mixed scheme proposed provides a uniform error estimate and avoids the membrane locking for the Naghdi shell under some geometrical restrictions. The idea of this pioneering paper has been widely reported in the literature [6,11,14,15,17,20,21].

For the Koiter shell, which is largely used for commercial codes and in the recent research [5,19,22], the problem of finite element approximation is twofold: numerical locking and complexity. The membrane locking is unavoidable and occurs with conformal or non conformal computations for bending-dominated shells [8] and still arouses the interest of researchers [19,22].

In this paper, the approach of Arnold and Brezzi is used with important changes and the same restrictions. We consider a Koiter shell. This completely changes the complexity of the approximation since variables of class C^1 appear in the formulation. In [20], the space H^2 is approximated by the Argyris element. This indicates, in addition to the complexity, a highly unbalanced combination of finite elements used for different variables. Here, we approach the membrane stresses by discontinuous piecewise polynomials, the tangential displacements by continuous piecewise polynomials augmented by

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bubbles and the transversal displacements by the *HCT* (Hsieh–Clough–Tocher) element expanded by C^1 bubbles. Its accuracy feature is guaranteed for less regularity and less numerical cost than the Argyris element which needs at least 18 degrees of freedom and locks for very thin shells [8,12,13]. The main aim of the paper is to design a mixed locking-free scheme for Koiter shell by using the conformal *HCT* element.

We prove an inf-sup discrete condition under the same geometric assumptions of [1]. We establish optimal error estimates independent of the thickness and then prove that the membrane locking is avoided. We finally make numerical results which confirm the theoretical predictions. For the hemicylindrical shell, we show the performance of the mixed scheme for small thickness. Some singularities appear for the hyperbolic paraboloid test which is not covered by the theoretical stability predictions.

The organization of the paper is as follows. In Section 1, we present the linear Koiter model. In Section 2, a mixed formulation is introduced and analyzed. In Section 3, a discretization strategy by finite elements is presented and uniform convergence is proved. In Section 4, we implement the mixed variational formulation and the primal variational formulation with FreeFem++ and then we compare their convergence sensitivity with respect to the thickness parameter on different examples.

2. The Koiter shell model

Greek indices take their values in the set {1, 2} and the Latin indices take their values in {1, 2, 3}. Products containing repeated indices are summed. Let ω be an open convex domain of $\mathbb{R}^2$. We consider a shell whose midsurface is given by $S=\varphi(\bar{\omega})$ where $\varphi\in C^3(\omega, R^3)$ is an injective mapping. Let $\vec{a}_\alpha=\varphi_{,\alpha},\ \alpha=1,2;\ \vec{a}_3=\frac{\vec{a}_1\wedge\vec{a}_2}{\|\vec{a}_1\wedge\vec{a}_2\|}$ be the covariant basis vectors and $\vec{a}^\alpha$ defined by $\vec{a}^\alpha.\vec{a}_\alpha=\delta^\alpha_\beta;\ \vec{a}_3=\vec{a}^3$ be the contravariant basis vectors. Let t be the shell thickness. The first and second fundamental forms of the midsurface are defined componentwise by

$$a_{\alpha\beta}=\vec{a}_\alpha.\vec{a}_\beta,\qquad b_{\alpha\beta}=\vec{a}_3.\vec{a}_{\alpha,\beta}=-\vec{a}_\alpha.\vec{a}_{3,\beta}.$$

Let $a=\|\vec{a}_1\wedge\vec{a}_2\|^2$ be the determinant of $(a_{\alpha\beta})_{\alpha\beta}$. We note $a^{\alpha\beta}=\vec{a}^\alpha.\vec{a}^\beta$ the first fundamental form contravariant components and $b^\alpha_\gamma=a^{\alpha\beta}b_{\beta\gamma}$ the mixed components of the second fundamental form.

For a displacement field $\vec{u}$, the linearized changes of the curvature tensor $\underline{\underline{\Upsilon}}=(\Upsilon_{\alpha\beta})_{\alpha,\beta}$ and of the membrane tensor $\underline{\underline{\Lambda}}=(\Lambda_{\alpha\beta})_{\alpha,\beta}$ read in covariant components [2,3]:

$$\begin{aligned}\Upsilon_{\alpha\beta}(\vec{u})&=u_{3/\alpha\beta}-b^\sigma_\alpha b_{\sigma\beta}u_3+b^\sigma_\alpha u_{\sigma/\beta}+b^\sigma_\beta u_{\sigma/\alpha}+b^\sigma_{\beta/\alpha}u_\sigma,\\ \Lambda_{\alpha\beta}(\vec{u})&=\tfrac{1}{2}(u_{\alpha/\beta}+u_{\beta/\alpha})-b_{\alpha\beta}u_3,\end{aligned}\tag{1}$$

where the covariant derivative of displacement components and second fundamental form mixed components is given by:

$$u_{\alpha/\beta}=u_{\alpha,\beta}-\Gamma^\delta_{\alpha\beta}u_\delta,\tag{2}$$

$$u_{3/\alpha\beta}=u_{3,\alpha\beta}-\Gamma^\delta_{\alpha\beta}u_{3,\delta},\tag{3}$$

$$b^\beta_{\alpha/\rho}=b^\beta_{\alpha,\rho}+\Gamma^\beta_{\rho\sigma}b^\sigma_\alpha-\Gamma^\sigma_{\alpha\rho}b^\beta_\sigma,\tag{4}$$

$$\Gamma^\delta_{\alpha\beta}=a^\delta.a_{\alpha\beta}.\tag{5}$$

Let $\underline{\underline{\underline{\underline{E}}}}=(E^{\alpha\beta\lambda\mu})_{\alpha\beta\lambda\mu}$ be the elasticity tensor, assumed to be elliptic as well as its inverse, given by $E^{\alpha\beta\lambda\mu}=\frac{\epsilon}{2(1+\nu)}(a^{\alpha\lambda}a^{\beta\mu}+a^{\alpha\mu}a^{\beta\lambda}+\frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu})$ where $\epsilon>0$ and $\nu\in(0,\frac{1}{2})$ are respectively the Young's modulus and Poisson ratio. We suppose the shell clamped on a part $\Gamma\neq\emptyset$ of its boundary and set

$V=\{\vec{v}=v_ia^i,\ v_\alpha\in H^1_\Gamma(\omega),\ v_3\in H^2_0(\omega)\}$,

$H^1_\Gamma(\omega)=\{u\in H^1(\omega),\ u=0\ \text{ on}\,\Gamma\}$,

$H^2_0(\omega)=\{u\in H^2(\omega),\ u=\frac{\partial u}{\partial n}=0\ \text{ on}\,\partial\omega\}$.

Note that V is a Hilbert space when endowed with the norm

$||\vec{v}||_V=(\sum_\alpha||v_\alpha||^2_{H^1}+||v_3||^2_{H^2})^{1/2}$.

Consider the bending-dominated Koiter shell problem

$$(\mathbf{P})\begin{cases}\quad\text{Find}\quad \vec{u}\in V\\ \tilde{A}(\vec{u};\vec{v})=\tilde{L}(\vec{v})\quad\forall\ \vec{v}\in V,\end{cases}\tag{6}$$

where $\tilde{L}$ is a continuous linear form corresponding to external forces and $\tilde{A}$ is a bilinear form corresponding to internal energy given by

$$\tilde{A}(\vec{u};\vec{v})=\frac{t^3}{12}\int_\omega E^{\alpha\sigma\lambda\mu}\Upsilon_{\alpha\sigma}(\vec{u})\Upsilon_{\lambda\mu}(\vec{v})\sqrt{a}dx+t\int_\omega E^{\alpha\sigma\lambda\mu}\Lambda_{\alpha\sigma}(\vec{u})\Lambda_{\lambda\mu}(\vec{v})\sqrt{a}dx.\tag{7}$$