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On Markov's theorem on zeros of orthogonal polynomials revisited

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ABSTRACT

This paper briefly surveys Markov's theorem related to zeros of orthogonal polynomials. Monotonicity of zeros of some families of orthogonal polynomials are reviewed in detail.

1. Introduction

Markov's theorem, dating back to the late 19th century, furnishes a method for obtaining information about zeros of orthogonal polynomials from the weight function related to orthogonality. Formally, adopting modern terminology, his result is stated as follows (see [22]):

Theorem 1.1 [22]. *Let $\{p_n(x, t)\}$ be a sequence of polynomials which are orthogonal on the interval $A = (a, b)$ with respect to the weight function $\omega(x, t)$ that depends on a parameter t, $t \in B = (c, d)$, i.e.,*

$$\int_a^b p_n(x,t)p_m(x,t)\omega(x,t)dx = 0, \quad m \neq n.$$

Suppose that $\omega(x, t)$ is positive and has a continuous first derivative with respect to t for $x \in A$, $t \in B$. Furthermore, assume that

$$\int_a^b x^k \frac{\partial \omega}{\partial t}(x,t)dx, \quad k = 0, 1, \ldots, 2n-1,$$

converge uniformly for t in every compact subinterval of B. Then the zeros of $p_n(x, t)$ are increasing (decreasing) functions of t, $t \in B$, provided that

$$\frac{1}{\omega(x,t)}\frac{\partial \omega}{\partial t}(x,t)$$

is an increasing (decreasing) function of x, $x \in A$.

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Markov's proof is based on the orthogonality relation (cf. [22, Eq. (2)]) together with the chain rule (cf. [22, Eq. (5)]), supposing that the zeros are defined implicitly as differentiable functions of the parameter. In addition, as an application of this result, Markov established that the zeros of Jacobi polynomials, which are orthogonal in $(-1, 1)$ with respect to the weight function $\omega(x, \alpha, \beta) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta > -1$, are decreasing functions of α and increasing functions of β. Later, in 1939, Szegő, in his classical book [26, Theorem 6.12.1, p. 115], provided a different proof of Markov's theorem. Szegő referred his proof of Theorem 1.1 in the following way [26, Footnote 31, p. 116]: *"This proof does not differ essentially from the original one due to A. Markov, although the present arrangement is somewhat clearer."* Szegő's reasoning (argument, approach) is based on Gauss mechanical quadrature, which was an approach that Stieltjes suggested to handle the problem, see [25, Section 5, p. 391]. In 1971, Freud (see [9, Problem 16, p. 133]) formulated a version of Markov's theorem that is a little more general, considering sequences of polynomials orthogonal with respect to measures in the form $d\alpha(x,t) = \omega(x,t)d\nu(x)$. A proof of a result appears in Ismail [12, Theorem 3.2, p. 183] (see also in Ismail's book [13, Theorem 7.1.1, p. 204]). Ismail's argument of the proof is also based on Gauss mechanical quadrature. As consequence, Ismail provides monotonicity properties for the zeros of Hahn and Meixner polynomials (see [13, Theorem 7.1.2, p. 205]). Kroó and Peherstorfer [18, Theorem 1], in a more general context of approximation theory, extended Markov's result to zeros of polynomials which have the minimal L_p-norm. Their approach is based on the implicit function theorem.

The main concern of this work derives from Markov's classic 1886 theorem. This allows the approach to be tailored towards measures with continuous and discrete parts, thus extending Markov's result. This point at issue was posed by Ismail in his book as an open problem [13, Problem 24.9.1, p. 660] (see also [12, Problem 1, p. 187]). The question is stated as follows:

Problem 1.1. Let μ be a positive and nontrivial Borel measure on a compact set $A \subset \mathbb{R}$. Assume that $d\mu(x, t)$ has the form

$$d\alpha(x,t) + d\beta(x,t), \tag{1.1}$$

where $d\alpha(x, t) := \omega(x, t)d\nu(x)$ and $d\beta(x,t) := \sum_{i=0}^{\infty} J_i(t)\delta_{y_i(t)}$,[1] with $t \in B$, B being an open interval on $\mathbb{R}$. Determine sufficient conditions in order for the zeros of the polynomial $P_n(x, t)$ to be strictly increasing (decreasing) functions of t.

The manuscript is organized in the following way: in Section 2 the main result is stated and proved; in Section 3 some conclusions are drawn from the main result, including Markov's classic theorem, among others; finally, in Section 4, illustrative examples are given: in Sections 4.1 and 4.2 monotonicity properties of zeros of polynomials orthogonal with respect measures with discrete parts are investigated; in Section 4.3, sharp monotonicity properties involving the zeros of Gegenbauer–Hermite, Jacobi–Laguerre and Laguerre–Hermite orthogonal polynomials are derived.

2. Main results

The next result extends Markov's theorem to measures with continuous and discrete parts, giving an answer to Problem 1.1. For a result in the context of polynomials which have minimal L_p-norm see [2, Theorem 1.1].

Theorem 2.1. *Assume the notation and conditions of Problem 1.1. Assume further the existence and continuity for each $x \in A$ and $t \in B$ of $(\partial\omega/\partial t)(x, t)$ and, in addition, suppose that*

$$G(t, x_1, \dots, x_n) := \sum_{i=0}^{\infty} g_i(t, x_1, \dots, x_n)$$

converges at $t = t_0$ and

$$\frac{\partial G}{\partial t}(t, x_1, \dots, x_n) := \sum_{i=0}^{\infty} \frac{\partial g_i}{\partial t}(t, x_1, \dots, x_n),$$

$$\frac{\partial G}{\partial x_j}(t, x_1, \dots, x_n) = \sum_{i=0}^{\infty} \frac{\partial g}{\partial x_j}(t, x_1, \dots, x_n),$$

converge uniformly for $t \in B$, where

$$g_i(t, x_1, \dots, x_n) = J_i(t)(y_i(t) - x_k)^{-1} \prod_{j=1}^{n} (y_i(t) - x_j)^2$$

and $(x_1, \dots, x_n) \in \mathbb{R}^n$. Denote by $x_1(t), \dots, x_n(t)$ the zeros of $P_n(x, t)$. Fix $k \in \{1, \dots, n\}$ and set

$$d_{k,i}(t) := \begin{cases} y_i(t) - x_k(t) & \text{if } y_i(t) \neq x_k(t), \\ 1 & \text{if } y_i(t) = x_k(t). \end{cases}$$

[1] The Dirac measure δ_y is a positive Radon measure whose support is the set $\{y\}$.