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Regular non-hamiltonian polyhedral graphs

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A B S T R A C T

Invoking Steinitz' Theorem, in the following a *polyhedron* shall be a 3-connected planar graph. From around 1880 till 1946 Tait's conjecture that cubic polyhedra are hamiltonian was thought to hold—its truth would have implied the Four Colour Theorem. However, Tutte gave a counterexample. We briefly survey the ensuing hunt for the smallest non-hamiltonian cubic polyhedron, the Lederberg-Bosák-Barnette graph, and prove that there exists a non-hamiltonian essentially 4-connected cubic polyhedron of order n if and only if $n \geq 42$. This extends work of Aldred, Bau, Holton, and McKay. We then present our main results which revolve around the quartic case: combining a novel theoretical approach for determining non-hamiltonicity in (not necessarily planar) graphs of connectivity 3 with computational methods, we dramatically improve two bounds due to Zaks. In particular, we show that the smallest non-hamiltonian quartic polyhedron has at least 35 and at most 39 vertices, thereby almost reaching a quartic analogue of a famous result of Holton and McKay. As an application of our results, we obtain that the shortness coefficient of the family of all quartic polyhedra does not exceed 5/6. The paper ends with a discussion of the quintic case in which we tighten a result of Owens.

1. Introduction

Due to Steinitz' classic theorem that the 1-skeleta of 3-polytopes are exactly the 3-connected planar graphs [34], we shall call such a graph a *polyhedron*. While the rigorous study of hamiltonian cycles goes back to at least 1766, when Euler treated the knight's tour problem, Hamilton and Kirkman were the first to study spanning cycles in polyhedra. We refer to [4] for further historical details. By Euler's formula, there are k-regular polyhedra for three values of k: 3, 4, or 5. We will call these *cubic*, *quartic*, and *quintic*, respectively. We use the word "regular" exclusively in the graph-theoretical sense of having all vertices of the same degree. The paper is split naturally into three sections—cubic, quartic, and quintic polyhedra—, and we give in each of these separately the motivation for treating the respective problem. Our results mainly revolve around the following six numbers: let c_k (p_k) denote the order of the smallest k-regular non-hamiltonian (non-traceable) polyhedron for $k \in \{3, 4, 5\}$.

For a possibly disconnected graph G, let $\omega(G)$ denote the number of connected components of G. In this article, a "cut" shall always be a vertex-cut, i.e. a vertex-set X in a graph G such that $\omega(G - X) \geq 2$. If we are referring to edge-cuts (defined analogously to vertex-cuts), we will explicitly mention this. We tacitly use the fact that for a 3-cut X in a polyhedron G, we have $\omega(G - X) = 2$, and that by Tutte's theorem [39] stating that 4-connected polyhedra are hamiltonian, a non-hamiltonian

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polyhedron contains at least one 3-cut. For more on the interplay between polyhedra, cuts, and hamiltonicity, we refer to the survey [32]. Let G be a polyhedron of connectivity 3 and $X = \{u, v, w\}$ a 3-cut in G. X is *trivial* if one of the components of $G - X$ is K_1. If G' is a component of $G - X$, then $G[V(G') \cup X]$ is called a *fragment* with *attachments* u, v, w (where $G[V(G') \cup X]$ denotes the subgraph of G induced by $V(G') \cup X$). A fragment F with attachments u, v, w is called an *ijk-fragment* if the degrees of u, v, w in F are i, j, k, respectively.

2. The cubic case

Tait conjectured [35] in 1884 that every cubic polyhedron is hamiltonian. This conjecture became famous because it implied the Four Colour Theorem (which at that time was itself open): by Jordan's Curve Theorem, any hamiltonian cycle $\mathfrak{h}$ in a cubic polyhedron G naturally divides the plane into an unbounded region A and a bounded region B with $A \cap B = \mathfrak{h}$. The duals of the planar graphs $G \cap A$ and $G \cap B$ are trees, so we may colour their vertices alternatingly. Thus, we can colour the faces of G with four colours such that no two adjacent faces receive the same colour. One can reduce the case of general polyhedra to cubic polyhedra, and thus, Tait's conjecture would have implied the Four Colour Theorem. However, Tait's conjecture turned out to be false and the first to construct a counterexample was Tutte in 1946, see [38], using in his approach three copies of a graph that has become to be known as "Tutte-fragment". The smallest counterexample is due to Lederberg (and independently, Bosák and Barnette), has order 38, and is also based on Tutte-fragments. (To be precise, there are six structurally very similar such graphs [19].) That this is indeed the smallest possible counterexample was shown by Holton and McKay [19] after a long series of papers by various authors, see for instance work of Butler [10], Barnette and Wegner [2], and Okamura [28]. The second part of the next theorem follows directly from the Holton–McKay result by successively substituting a vertex with a triangle.

Theorem 1 (Holton and McKay [19]). *We have that*

$$c_3 = 38.$$

Furthermore, for each even $n \geq 38$ there is a non-hamiltonian cubic polyhedron on n vertices.

Balinski asked whether non-traceable cubic polyhedra exist. Brown and independently Grünbaum and Motzkin proved the existence of such graphs. Klee asks for determining p_3. (We refer to Klee's excellent [13, Chapter 17] for references and further details.) The best bounds that are known are as follows.

Theorem 2 (Knorr [23] and T. Zamfirescu [44]). *We have that*

$$54 \leq p_3 \leq 88.$$

Furthermore, for each even $n \geq 88$ there exists a non-traceable cubic polyhedron of order n.

The lower bound was proven by Knorr in 2010, see [23]. The upper bound due to Zamfirescu [44], although published in 1980, has resisted all improvement attempts hitherto. Knorr's lower bound is based on work of Okamura [28] and improves a result of Hoffmann [18], while the upper bound is based on Tutte-fragments and improves a result of Brown [13, p. 362]. That indeed there exists a non-traceable cubic polyhedron of order n for every even $n \geq 88$ can be shown again by simply replacing vertices with triangles.

Recently, McKay raised the question [25] which plane graphs have a spanning tree such that at least one edge of each face is in the tree, specifically asking whether triangulations always have such a tree. Following an argument of Kynčl [25], a triangulation has a spanning tree with the required property if and only if its dual is traceable. Thus, every graph providing a positive answer to Balinski's question yields an example for a negative answer to McKay's question, and vice-versa.

A 3-connected graph is *essentially 4-connected* if all of its 3-cuts are trivial. Essentially 4-connected k-regular polyhedra that are not 4-connected only exist for $k = 3$. Let c_3^* (p_3^*) be the order of the smallest non-hamiltonian (non-traceable) essentially 4-connected cubic polyhedron. We shall see that c_3^* is known. However, we first give a definition and a well-known lemma (see for instance [12]), the proof of which we include for completeness' sake.

In a graph G, a set S of k edges is called a *k-edge-cut* if $G - S$ is disconnected and if no proper subset of S satisfies this property. It is easy to see that $\omega(G - S) = 2$. These two components are called *k-pieces*. A k-edge-cut is called *non-trivial* if each of its k-pieces contains a cycle. We say that a cubic graph is *cyclically k-edge-connected* if it has no non-trivial t-edge-cuts for $0 \leq t \leq k - 1$. For $X, Y \subset V(G)$ we denote with $E(X, Y)$ the set of all edges with one endpoint in X and the other endpoint in Y. Abusing notation, when $X = \{x\}$, we shall write $E(x, Y)$ instead of $E(\{x\}, Y)$. For a vertex $v \in V(G)$ and a subgraph H of G, we denote with $N_H(v)$ the set of neighbours of v in H, and put $N_G(v) = N(v)$.

Lemma 1. *A 3-connected cubic graph of order $\neq 6$ is essentially 4-connected if and only if it is cyclically 4-edge-connected.*

Proof. Assume G is essentially 4-connected, but not cyclically 4-edge-connected. Then there is a 3-edge-cut M such that each 3-piece contains a cycle. Since $G \not\cong K_4$ and $|V(G)| \neq 6$, we have that $|V(G)| \geq 8$, so one of these pieces H (say) contains at least four vertices. If $|V(H)| = 4$, then its sum of vertex-degrees would be odd (namely 9), which is impossible. So $|V(H)| \geq 5$. Taking the endpoints of the edges in M lying in H, we obtain a non-trivial k-cut with $k \leq 3$, a contradiction.